Department of EECS - University of California at Berkeley
EECS126 - Probability and Random Processes - Spring 2003
Midterm No. 2: 4/08/2003 - Solutions

Problem 1 (10%). Give an example of a pair of random variables (X, Y) that are uncorrelated and not independent.

For instance, let (X, Y) that takes the four values $\{(-1, 0), (0, -1), (1, 0), (0, 1)\}$ with equal probabilities. Then $E(X) = 0, E(Y) = 0, E(XY) = 0$, so that $E(XY) = E(X)E(Y)$ and the random variables X, Y are uncorrelated. However, $P(X = 1, Y = 1) = 0$ whereas $P(X = 1) = 1/4$ and $P(Y = 1) = 1/4$. Hence, $P(X = 1, Y = 1) \neq P(X = 1)P(Y = 1)$, which shows that the random variables X, Y are not independent.

Problem 2 (10%). Give an example of a pair of random variables (X, Y) that are not independent and are such that $E[X|Y] = E(X)$.

The example we gave for Problem 1 meets that condition. Indeed, $E[X|Y = -1] = 0, E[X|Y = 0] = 0, E[X|Y = 1]$, so that $E[X|Y] = 0 = E(X)$.

Problem 3 (10%). Is it possible for a pair of random variables (X, Y) to be such that $E[X|Y] > X$ for all Y? Explain your answer.

No, this is not possible. We know that $E(E[X|Y]) = E(X)$. However, if it were the case that $E[X|Y] > X$, then we would conclude that $E(E[X|Y]) > E(X)$, a contradiction.

Problem 4 (10%). Let X, Y, Z be independent and uniformly distributed on $[-1, 1]$. Calculate $E[X + Y|X + Y + Z]$.

By symmetry,

$$E[X + Y|X + Y + Z] = E[Y + Z|X + Y + Z] = E[X + Z|X + Y + Z].$$

If we designate the random variable above by V, then we see by adding all the three terms that

$$3V = E[2X + 2Y + 2Z|X + Y + Z] = 2(X + Y + Z).$$

Hence, $E[X + Y|X + Y + Z] = V = 2(X + Y + Z)/3$.

Problem 5 (15%). Let X, Y, Z be independent and equally likely to take the values $\{-2, -1, 0, 1, 2\}$. Calculate $L[X + 2Y|X + Y, Y + Z]$.

Let $U = X + 2Y, V_1 = X + Y, V_2 = Y + Z$. We know that

$$L[U|V] = \Sigma_{U,V}\Sigma_V^{-1}V.$$

Now,

$$\Sigma_{U,V} = E((U(V_1, V_2))) = [3a, 2a] \text{ where } a = E(X^2) = E(Y^2) = E(Z^2)$$

and

$$\Sigma_V = E(V(V_1, V_2)) = \begin{bmatrix} 2a & a \\ a & 2a \end{bmatrix},$$

so that

$$\Sigma_V^{-1} = \frac{1}{3a}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Hence,

$$L[U|V] = a[3, 2]\frac{1}{3a}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} V = [\frac{4}{3}, \frac{2}{3}]V = \frac{4}{3}(X + Y) + \frac{2}{3}(Y + Z).$$

Problem 6 (25%). Let X, Z be independent with $P(X = 0) = 0.4, P(X = 1) = 0.6$, and $Z = N(0, 1)$. Find the MLE and the MAP of X given $Y = X + (1 + X)Z$.

MLE: Let

$$L(Y) = \frac{f_{Y|X}[y|1]}{f_{Y|X}[y|0]}.$$

We see that when $X = 1, Y = N(1, 4)$ and when $X = 0, Y = N(0, 1)$. Hence

$$f_{Y|X}[y|1] = \frac{1}{\sqrt{8\pi}} \exp\{-\frac{1}{8}(y-1)^2\}$$

and

$$f_{Y|X}[y|0] = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}y^2\}.$$

Consequently,

$$L(y) = \frac{1}{2} \exp\{\frac{3}{8}y^2 + \frac{1}{4}y - \frac{1}{8}\}.$$

Since $MLE[X|Y = y] = 1\{L(y) \geq 1\}$, we conclude that

$$MLE[X|Y = y] = \begin{cases} 0, & \text{if } y \in (\frac{4-\sqrt{19}}{3}, \frac{4+\sqrt{19}}{3}) \\ 1, & \text{otherwise.} \end{cases}$$

MAP: We find that for $x \in \{0, 1\}$,

$$P[X = x|Y = y] = \frac{P(X = x) f_{Y|X}[y|x]}{f_Y(y)}.$$

Hence,

$$\begin{aligned} MAP[X|Y = y] &= 1\{P[X = 1|Y = y] \geq P[X = 0|Y = y]\} \\ &= 1\{L(y) \geq \frac{P(X = 0)}{P(X = 1)}\} = 1\{L(y) \geq \frac{2}{3}\}. \end{aligned}$$

Consequently,

$$MLE[X|Y = y] = \begin{cases} 0, & \text{if } y \in (\frac{4}{3} - \sqrt{\frac{19}{3} - \frac{8}{3}\ln(\frac{2}{3})}, \frac{4}{3} + \sqrt{\frac{19}{3} - \frac{8}{3}\ln(\frac{2}{3})}) \\ 1, & \text{otherwise.} \end{cases}$$

Problem 7 (30%). For $x = 0, 1$, given $X = x$, Y is exponentially distributed with mean $\mu(x)$, for $x = 0, 1$ where $0 < \mu(0) < \mu(1)$.

a. Find $\hat{X} = g(Y)$ that maximizes $P[\hat{X} = 1 | X = 1]$ subject to $P[\hat{X} = 1 | X = 0] \leq 5\%$.

b. Assume that $\mu(0) = 1$. Find the minimum value of $\mu(1)$ so that $P[\hat{X} = 1 | X = 1] \geq 95\%$.

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a. We know that $\hat{X} = 1\{L(Y) \geq \lambda\}$ where λ is such that $P[\hat{X} = 1 | X = 0] = 5\%$. Now, with $\lambda(x) := \mu^{-1}(x)$,

$$L(y) = \frac{f_{Y|X}[y|1]}{f_{Y|X}[y|0]} = \frac{\lambda(1) \exp\{-\lambda(1) y\}}{\lambda(0) \exp\{-\lambda(0) y\}}.$$

Hence, $\hat{X} = 1\{y \geq y_0\}$ where y_0 is such that $P[\hat{X} = 1 | X = 0] = 5\%$. That is,

$$5\% = P[Y \geq y_0 | X = 0] = \exp\{-\lambda(0) y_0\},$$

i.e.,

$$y_0 = -\frac{\ln(0.05)}{\lambda(0)}.$$

b. In this case, $y_0 = \ln(20)$. Consequently,

$$P[\hat{X} = 1 | X = 1] = P[Y \geq y_0 | X = 1] = \exp\{-\lambda(1) y_0\}.$$

Hence, we want

$$95\% = \exp\{-\lambda(1) y_0\} = \exp\{-\lambda(1) \ln(20)\} = (20)^{-\lambda(1)},$$

so that

$$\ln(0.95) = -\lambda(1) \ln(20), \text{ or } \lambda(1) = -\frac{\ln(0.95)}{\ln(20)},$$

which gives

$$\mu(1) = -\frac{\ln(20)}{\ln(0.95)}.$$