Department of EECS - University of California at Berkeley
EECS126 - Probability and Random Processes - Spring 2000
Midterm No. 1: 2/23/2000

SOLUTIONS

Problem 1: You flip a fair coin repeatedly. What is the probability that you have to flip it exactly 10 times to see two "heads"?

Solution:

There must be exactly one head among the first nine flips and the last flip must be another head. The probability of that event is

$$9(\frac{1}{2})^9 \times (\frac{1}{2}) = \frac{9}{2^{10}}.$$

Problem 2: Let A, B, C be three events. Assume that $P(A) = 0.6, P(B) = 0.6, P(C) = 0.7, P(A \cap B) = 0.3, P(A \cap C) = 0.4, P(B \cap C) = 0.4$, and $P(A \cup B \cup C) = 1$. Find $P(A \cap B \cap C)$.

Solution:

We know that (draw a picture)

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

Substituting the known values, we find

$$1 = 0.6 + 0.6 + 0.7 - 0.3 - 0.4 - 0.4 + P(A \cap B \cap C),$$

so that

$$P(A \cap B \cap C) = 0.2.$$

Problem 3: There are two coins. The first coin is fair. The second coin is such that $P(H) = 0.6 = 1 - P(T)$. You are given one of the two coins, with equal probabilities between the two coins. You flip the coin four times and three of the four outcomes are H. What is the probability that your coin is the fair one?

Solution:

Let A designate the event "your coin is fair." Let also B designate the event "three of the fair outcomes are H."

We know that

$$\begin{aligned} P[A|B] &= \frac{P(A \cap B)}{P(A)} = \frac{P[B|A]P(A)}{P[B|A]P(A) + P[B|A^c]P(A^c)} \\ &= \frac{C(4,3)(1/2)^4}{C(4,3)(1/2)^4 + C(4,3)(0.6)^3(0.4)} = \frac{2^{-4}}{2^{-4} + (0.6)^3 0.4}. \end{aligned}$$

Problem 4: Define the random variable X as follows. You throw a dart uniformly in a circle with radius 5. The random variable X is equal to 2 minus the distance between the dart and the center of the circle if this distance is less than or equal to one. Otherwise, X is equal to 0.

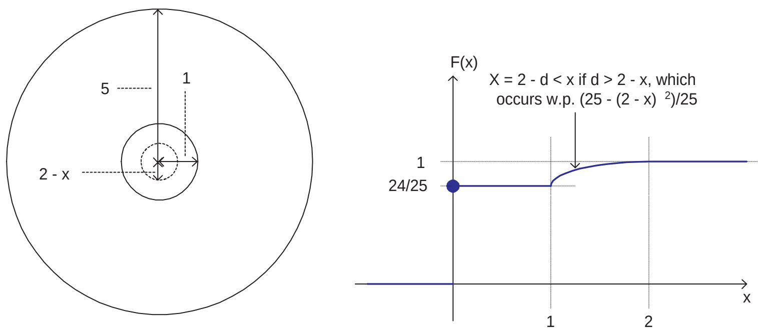
a. Plot carefully the probability distribution function $F(x) = P(X \leq x)$ for $x \in \Re := (-\infty, +\infty)$.

b. Give the mathematical expression for the probability density function $f(x)$ of X for $x \in \Re := (-\infty, +\infty)$.

Solution:

Let Y be the distance between the dart and the center of the circle.

a. When $1 \leq x \leq 2$, $X \leq x$ if $Y \geq 2 - x$, which occurs with probability $(25 - (2-x)^2)/25$. Also, $X = 0$ if $Y > 1$, which occurs with probability $(25-1)/25 = 24/25$. These observations translate into the plot shown below:



b. Taking the derivative of $F(x)$, one finds

$$f(x) = \frac{24}{25}\delta(x) + \frac{2x-4}{25}1\{1 < x < 2\}.$$