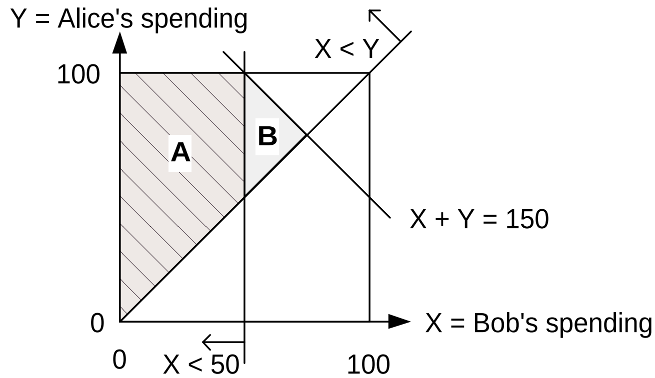
Department of EECS - University of California at Berkeley
EECS126 - Probability and Random Processes - Spring 2000
Final Examination: 5/15/2000

SOLUTIONS

Problem 1: Bob and Alice go Christmas shopping and agree to spend independent amounts uniformly distributed between 0 and 100. What it the probability that Bob spent less than 50 given that together they spent less than 150 and that Bob spent less than Alice?



The figure shows the sample space and the various possible inequalities. The requested conditional probability is the ratio of the area of A over the sum of the areas of A and B. The areas of A and B are

$$area(A) = \frac{1}{2}(100 + 50) \times 50 = 3750$$

and

$$area(B) = \frac{1}{4}50^2 = 625.$$

Hence, the requested probability is equal to $\frac{3750}{3750+625} \approx 0.86$.

Problem 2: Let $\{X_n, n \geq 0\}$ be independent standard Gaussian random variables. Define

$$Y_{n+1} = aY_n + bX_n, n \geq 0$$

where Y_0 is $N(0, \sigma^2)$ and independent of $\{X_n, n \geq 0\}$ and $a \in (-1, 1), b \neq 0$ are known numbers.

a. Find σ^2 so that the probability distribution function of Y_n does not depend on $n \geq 0$.
b. Calculate $E[Y_n|Y_0]$.

a. Taking the square of both sides of the equation that defines Y_n, we find

$$Y_{n+1}^2 = a^2Y_n^2 + b^2X_n^2 + 2abY_nX_n.$$

Taking expectations and using stationarity of $\{Y_n\}$ and the independence of Y_n and X_n, we find that

$$\sigma^2 = a^2\sigma^2 + b^2, \text{ so that } \sigma^2 = \frac{b^2}{1-a^2}.$$

b. We have

$$Y_n = a^n Y_0 + \sum_{m=0}^{n-1} a^{n-1-m} b X_m,$$

so that

$$E[Y_n | Y_0] = a^n Y_0.$$

Problem 3: A power line is made of N parallel lines. Each line is down with probability $p \in (0,1)$, independently of the others. When a line is up, it can carry a current of 1 Ampere. When one line is down, the whole power line is shut down for repair. Find the value of N that maximizes the average power that the line carries.

The power that the line carries is N Amperes with probability $(1-p)^N$. Otherwise, the line does not carry any current. Consequently, the expected power is

$$f(N) = N(1-p)^N.$$

It is now a matter of optimizing over N. Let N^* be the best value of N. Then, $f(N^*-1) \leq f(N^*)$ and $f(N^*+1) \leq f(N^*)$. The first inequality gives (with $q := 1-p$ to simplify the notation)

$$(N^*-1)q^{N^*-1} \leq N^* q^{N^*}, \text{ or, } N^* \leq \frac{1}{p}.$$

The second inequality gives

$$(N^*+1)q^{N^*+1} \leq N^* q^{N^*}, \text{ or, } N^* \geq \frac{1}{p} - 1.$$

These inequalities show that $N^* \approx 1/p$.

Problem 4: Let X be a random variable uniformly distributed in $[0, 2\pi]$. Find the estimator $Z = a + bX$ that minimizes $E(\sin(X) - Z)^2)$.

By definition, we are looking for the LLSE of $Y = \sin(X)$ given X. We know that

$$L[Y|X] = E(Y) + \frac{cov(X,Y)}{vax(X)}(X - E(X)).$$

Now, $E(Y) = 0$ by symmetry; $E(X) = \pi$; $var(X) = E(X^2) - (E(X))^2 = 4\pi^2(1/3 - 1/4) = \pi^2/3$;

$$\begin{aligned}
cov(X,Y) &= E(XY) - E(X)E(Y) = E(X\sin(X)) = \frac{1}{2\pi}\int_0^{2\pi} x\sin(x)dx \\
&= -\frac{1}{2\pi}\int_0^{2\pi} x d\cos(x) = [-\frac{1}{2\pi}x\cos(x)]_0^{2\pi} + \frac{1}{2\pi}\int_0^{2\pi}\cos(x)dx \\
&= -1 + \frac{1}{2\pi}[\sin(x)]_0^{2\pi} = -1.
\end{aligned}$$

Putting these values together, we find

$$L[Y|X] = -\frac{1}{\pi^2/3}(X - \pi) = \frac{3}{\pi^2}(\pi - X).$$

Problem 5: For $n \geq 1$, let X_n be a Poisson random variable with mean n. Use the CLT to calculate

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!}.$$

We can write

$$e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = P(X_n \leq n) = P(\frac{X_n - n}{\sqrt{n}} \leq 0).$$

Now, for $n \gg 1$, $\frac{X_n - n}{\sqrt{n}} \approx N(0,1)$. Therefore,

$$P(\frac{X_n - n}{\sqrt{n}} \leq 0) \to \frac{1}{2} \text{ as } N \to \infty.$$

Problem 6: Let X, Y, Z be independent and $N(0,1)$. Calculate

$$E[X|aX + Y, aX + Z].$$

This is all standard. Let $V = aX + Y$ and $W = aX + Z$. Let also $U = (V, W)^T$. We have

$$E[X|U] = E(X) + \Sigma_{X,U}\Sigma_{U,U}^{-1}(U - E(U)) = \Sigma_{X,U}\Sigma_{U,U}^{-1}U.$$

Now,

$$\Sigma_{X,U} = E(XU^T) = E(X[aX + Y, aX + Z]) = (a, a).$$

Also,

$$\Sigma_{U,U} = E(UU^T) = E(\begin{bmatrix} aX + Y \\ aX + Z \end{bmatrix} [aX + Y, aX + Z]) = \begin{bmatrix} a^2 + 1 & a^2 \\ a^2 & a^2 + 1 \end{bmatrix}),$$

so that

$$\Sigma_{U,U}^{-1} = \frac{1}{2a^2 + 1} \begin{bmatrix} a^2 + 1 & -a^2 \\ -a^2 & a^2 + 1 \end{bmatrix}).$$

Substituting these values in the expression for $E[X|U]$, we find

$$E[X|V, W] = \frac{a}{2a^2 + 1}(V + W).$$

Problem 7: Let $\{X_n, n \geq 0\}$ be a discrete time Markov chain on the state space $\{1, 2, 3, 4\}$ with the following transition probability matrix:

$$P = \begin{bmatrix} 0 & 0 & 0.5 & 0.5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

a. Is this Markov chain irreducible?
b. Is the Markov chain aperiodic?

c. Find the stationary distribution of $\{X_n, n \geq 0\}$.
d. Calculate $E[T_4|X_0 = 3]$ where $T_4 = \min\{n \geq 0|X_n = 4\}$.

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a. Yes. The graph shows that the Markov chain can go from every state to every other state.
b. The Markov chain is periodic with period 3. For instance,

$$d(1) = \text{g.c.d}\{n \geq 1|P^n(1,1) > 0\} = \text{g.c.d}\{3, 6, 9, \ldots\} = 3.$$

c. We must solve the balance equations:

$$\pi(1) = \pi(2); \pi(2) = \pi(3) + \pi(4); \pi(3) = \pi(1)/2; \pi(4) = \pi(1)/2.$$

These equations show that $\pi(1) = \pi(2) = \alpha$, say, and $\pi(3) = \pi(4) = \alpha/2$. Since $\pi(1)+\pi(2)+\pi(3)+\pi(4) = 1$, one finds that $\alpha + \alpha + \alpha/2 + \alpha/2 = 1$, so that $\alpha = 1/3$. Hence,

$$\pi(1) = \pi(2) = 1/3 \text{ and } \pi(3) = \pi(4) = 1/6.$$

d. We write down the first step equations for $\beta(i) := E[T_4|X_0 = i]$:

$$\begin{aligned}
\beta(1) &= 1 + 0.5\beta(3) + 0.5 \times 0 \\
\beta(2) &= 1 + \beta(1) \\
\beta(3) &= 1 + \beta(2) \\
\beta(4) &= 0
\end{aligned}$$

From the third and second equations we find that

$$\beta(3) = 1 + \beta(2) = 2 + \beta(1).$$

Substituting this expression in the first equation we find that

$$\beta(1) = 1 + 0.5(2 + \beta(1)) = 2 + 0.5\beta(1).$$

Solving for $\beta(1)$ we find

$$\beta(1) = 4.$$

Using $\beta(3) = 2 + \beta(1)$ we conclude that

$$\beta(3) = E[T_4|X_0 = 3] = 6.$$

Problem 8 (10pts/100) Let X, Y be independent and exponentially distributed with mean 1. Given X and Y, the random variables $\{Z_n, n \geq 1\}$ are i.i.d. and uniformly distributed on $[X, X+Y]$.

Show that $(min\{Z_1, \ldots, Z_n\}, max\{Z_1, \ldots, Z_n\})$ are sufficient statistics for estimating X and Y given $\{Z_1, \ldots, Z_n\}$.

Let $U = \{Z_1, \ldots, Z_n\}, V = min\{Z_1, \ldots, Z_n\}$, and $W = max\{Z_1, \ldots, Z_n\}$. Let also $v = min\{z_1, \ldots, z_n\}$ and $w = max\{z_1, \ldots, z_n\}$.

We first observe that

$$
\begin{aligned}
f_{U|(X,Y)}[u|x,y] &= \Pi_{k=1}^{n} \frac{1\{x \leq z_k \leq x+y\}}{y} \\
&= \frac{1}{y^n} 1\{x \leq v \text{ and } w \leq x+y\} \\
&= h(x,y;v,w),
\end{aligned}
$$

for some function h. Consequently,

$$
f_{(X,Y)|U}[x,y|u] = \frac{f_{X,Y}(x,y) f_{U|(X,Y)}[u|x,y]}{f_U(u)} = \frac{f_{X,Y}(x,y) h(x,y;v,w)}{r(v,w)}
$$

where $r(v,w)$ is the integral of $f_{X,Y}(x,y)h(x,y;v,w)$ over all possible values of (x,y). We conclude that

$$
f_{(X,Y)|U}[x,y|u] = g(x,y;v,w)
$$

which proves that (V,W) is sufficient to estimate (X,Y) given U.