UNIVERSITY OF CALIFORNIA

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EECS 126 — MIDTERM #2 Solutions

1a. (i) Yes,

$$\begin{aligned} E(X+Y) &= \iint (x+y) f_{XY}(x,y)\,dx\,dy \\ &= \iint x f_{XY}(x,y)(dx)\,dy + \iint y f_{XY}(x,y)\,dx\,dy \\ &= E(X) + E(Y) \end{aligned}$$

(ii) No, true if X, Y are independent (and more generally, uncorrelated).

(iii) No, true if $Y = c$, a constant.

(iv) No, $E(XY|Y=3) = 3E(X|Y=3)$. This equals $3E(X)$ if, for example, X and Y are independent.

1b. (i) Yes,

$$\begin{aligned} E(Y|X=x) &= \int y f_{Y|X}(y|x)\,dy \\ &= \int y f_Y(y)\,dy = E(Y) \end{aligned}$$

for any x.

(ii)

$$\begin{aligned} E(Y|X) &= \frac{1}{2}E(X|X) + \frac{1}{2}E(-X|X) \\ &= 0 \\ &= E(Y) \end{aligned}$$

X and Y are clearly not independent.

2. Let X_i, Y_i be the horizontal and vertical displacement travelled at step i. Let D be the squared distance after n steps.

$$D = \left(\sum_{i=1}^{n} X_i\right)^2 + \left(\sum_{i=1}^{n} Y_i\right)^2$$

X_i, X_j are independent, $E(X_i) = 0$, and similarly for Y_i 's.

$$\text{So, } E(D) = \sum_{i=1}^{n} E(X_i^2) + \sum_{i \neq j} E(X_i X_j) + \sum_{i=1}^{n} E(Y_i^2) + \sum_{i \neq j} E(Y_i Y_j)$$

$$= \sum_{i=1}^{n} E(X_i^2) + \sum_{i=1}^{n} E(Y_i^2)$$

$$= \sum_{i=1}^{n} E(X_i^2 + Y_i^2)$$

$$= n$$

3a. $$E(N) = E[E(N|M)]$$

$$= \sum_{m=0}^{\infty} E(N|M=m)P(M=m)$$

$$= \sum_{m=0}^{\infty} m(1-\varepsilon)P(M=m)$$

$$= (1-\varepsilon)E(M) = \frac{(1-\varepsilon)(1-p)}{p}$$

3b. $$E(N^2) = E[E(N^2|M)]$$

$$E(N^2|M) = \mathrm{Var}(N^2|M) + [E(N|M)]^2$$

$$= M\varepsilon(1-\varepsilon) + [M(1-\varepsilon)]^2$$

$$E(N^2) = E[M\varepsilon(1-\varepsilon) + M^2(1-\varepsilon)^2]$$

$$= \frac{(1-p)\varepsilon(1-\varepsilon)}{p} + \left[\frac{1-p}{p^2} + \frac{(1-p)^2}{p^2}\right](1-\varepsilon)^2$$

$$= \frac{\varepsilon(1-\varepsilon)(1-p)}{p} + \frac{(1-p)(2-p)}{p^2}(1-\varepsilon)^2$$

3c. $$P(N=n) = \sum_{m=0}^{\infty} P(M=m)P(N=n|M=m)$$

$$P(N=n|M=m) = \begin{cases} \binom{m}{n}(1-\varepsilon)^n \varepsilon^{m-n} & \text{if } n \leq m \\ 0 & \end{cases}$$

$$\text{So, } P(N=n) = \sum_{m=n}^{\infty} \binom{m}{n}(1-\varepsilon)^n \varepsilon^{m-n} \cdot (1-p)^m p$$