EE126: Probability and Random Processes **F'10**

Midterm — October 26

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SOLUTIONS

Formulas: Given the short attention span induced by twitter and the like, we thought you might appreciate not having to remember the following formulas. After all, they are on Wikipedia.

$$\mathbf{X} = N(\mu, \Sigma) \Leftrightarrow f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\{-\frac{1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)\}$$

$$(\mathbf{X}, \mathbf{Y}) J.G. \Rightarrow E[\mathbf{X}|\mathbf{Y}] = E(\mathbf{X}) + \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{Y}}^{-1}(\mathbf{Y} - E(\mathbf{Y}))$$

$$\text{cov}(\mathbf{AX}, \mathbf{BY}) = \mathbf{A}\text{cov}(\mathbf{X}, \mathbf{Y})\mathbf{B}^T.$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $V = N(0,1)$, then $P(V > 1) = 0.159, P(V > 1.64) = 0.05,$
$P(V > 1.96) = 0.025, P(V > 2) = 0.023, P(V > 2.58) = 0.005.$

Problem 1. *(***Short Problems 40%***)*

- *Give an example where $E[X|Y] = E(X)$ but X, Y are not independent.*

  Let $X = YZ$ where Y, Z are independent and uniform in $[-1, 1]$.

- *Let X, Y be i.i.d., $B(100, 0.3)$. Calculate $E[X - Y|X + Y]$.*

  By symmetry, $E[X - Y|X + Y] = 0$.

- *Let X, Y be as in the previous problem. Calculate $E[(X + Y)^2|X]$.*

  We have

$$E[(X+Y)^2|X] = X^2 + 2XE(Y) + E(Y^2) = X^2 + 60X + \text{var}(Y) + E(Y)^2$$
$$= X^2 + 60X + 100 \times 0.3 \times 0.7 + (30)^2$$

- *Assume that* $\Sigma_X = \begin{bmatrix} 4 & -1 & -2 \\ -1 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix}$. *Calculate* $\text{var}(2X_1 + 3X_2 + X_3)$.

  One has $\text{var}(\mathbf{a}^T\mathbf{X}) = \mathbf{a}^T\Sigma_X\mathbf{a}$. *So,*

$$\text{var}([2,3,1]\mathbf{X}) = [2,3,1]\begin{bmatrix} 4 & -1 & -2 \\ -1 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = [3,4,-3]\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 13.$$

- *Let* X, Y, Z *be i.i.d.* $U[0,1]$. *Calculate* $E[2X + 3Y + 4Z | X + Y + Z]$.

  By symmetry,

$$E[X|X+Y+Z] = E[Y|X+Y+Z] = E[Z|X+Y+Z] = \frac{1}{3}(X+Y+Z).$$

  Hence,

$$E[2X+3Y+4Z|X+Y+Z] = (2+3+4) \times \frac{1}{3}(X+Y+Z) = 3(X+Y+Z).$$

Problem 2. *(20%) Assume that $Y = X + aZ$ where X, Z are i.i.d., N(0, 1).*

(a) Calculate $E[X|Y]$;

(b) What is the variance of X given Y?

(c) Given $Y = y$, what is the distribution of X?

(d) Express $P[X > c|Y = y]$ in terms of $\Phi(z) := P(Z \leq z)$ where $Z = N(0, 1)$.

(a) $E[X|Y] = \Sigma_{X,Y}\Sigma_Y^{-1}Y = Y/(1 + a^2) =: bY$.

(b) We know that $X - E[X|Y] = X - bY = X - b(X + aZ) = (1 - b)X - abZ$, so that the variance of X given Y is equal to

$$\sigma^2 = \text{var}((1 - b)X - abZ) = (1 - b)^2 + (ab)^2 = \frac{a^2}{1 + a^2}.$$

(c) Given $Y = y$, $X = N(by, \sigma^2)$.

(d) We have

$$P[X > c|Y = y] = P(N(by, \sigma^2) > c) = P(N(0, \sigma^2) > c - by)$$
$$= P(N(0, 1) > \frac{c - by}{\sigma}) = 1 - \Phi(\frac{c - by}{\sigma}).$$

Problem 3. *(20%) Assume that $Y = 4X + Z$ where $Z = N(0,1)$ and $P(X = k) = 1/3$ for $k \in \{-1, 0, 1\}$.*

a) Calculate $\hat{X} = MAP[X|Y]$.

b) Calculate $P(\hat{X} \neq X)$.

a) A sketch of the densities shows that

$$MAP[X|Y] = \begin{cases} -1, & \text{if } Y \leq -2 \\ 0, & \text{if } Y \in (-2, 2) \\ 1, & \text{if } Y \geq 2. \end{cases}$$

b) We find,

$$\begin{aligned} P(\hat{X} \neq X) &= \frac{1}{3}P[\hat{X} \neq X)|X = -1] + \frac{1}{3}P[\hat{X} \neq X)|X = 0] + \frac{1}{3}P[\hat{X} \neq X)|X = 1] \\ &= \frac{1}{3}P[Y \geq -1|X = -1] + \frac{1}{3}P[Y < -1 \text{ or } Y > 1|X = 0] + \frac{1}{3}P[Y < 1|X = 1] \\ &= \frac{1}{3}P(Z \geq 2) + \frac{1}{3}P(Z < -2 \text{ or } Z > 2) + \frac{1}{3}P(Z < -2) \\ &= \frac{1}{3}[\alpha + 2\alpha + \alpha] = \frac{4}{3}\alpha \end{aligned}$$

where $\alpha = P(Z > 2) = 0.023$. Thus,

$$P(\hat{X} \neq X) \approx 3\%.$$

Problem 4. *(**20%**) When $X = 0$, $Y = N(0, 1)$. When $X = 1$, $Y = N(0, 4)$.*

(a) Find $\hat{X} = g(Y)$ that maximizes $P[\hat{X} = 1|X = 1]$ subject to $P[\hat{X} = 1|X = 0] \leq 5\%$.

(b) What is $P[\hat{X} = 1|X = 1]$?

(a) One has

$$\hat{X} = 1 \text{ iff } L(y) \geq \lambda$$

where λ is such that $P[\hat{X} = 1|X = 0] = 5\%$. Now,

$$L(y) = \frac{f_1(y)}{f_0(y)} \text{ with } f_1(y) = \frac{1}{\sqrt{2\pi 4}} \exp\{-y^2/8\}, f_0(y) = \frac{1}{\sqrt{2\pi}} \exp\{-y^2/2\}.$$

Hence,

$$L(y) = A \exp\{y^2(\frac{1}{2} - \frac{1}{8})\} = A \exp\{3y^2/8\}.$$

Thus,

$$L(y) \geq \lambda \Leftrightarrow |y| > \alpha, \text{ for some } \alpha.$$

We should choose α so that

$$P[\hat{X} = 1|X = 0] = P[|Y| > \alpha|X = 0] = P(|N(0, 1)| > \alpha) = 5\%.$$

Hence, $\alpha = 1.96$.

(b) One finds

$$P[\hat{X} = 1|X = 1] = P[|Y| > \alpha|X = 1] = P(|N(0, 4)| > 1.96) \approx P(|N(0, 1)| > 1) \approx 32\%.$$