EE126: Probability and Random Processes **F'10**

Midterm — September 23

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Problem 1. (**Short Problems 25%**)

- *Give an example of two events A and B such that $P[A|B] = P(B) > P(A)$.*

  Consider the uniform distribution on $[0,1]$. *Let* $B = [0, 1/2]$ *and* $A = [1/4, 5/8]$. *Then,* $P[A|B] = P(A \cap B)/P(B) = P([1/4, 1/2])/P(B) = 1/2 = P(B)$ *and* $P(A) = 3/8 < P(B)$.

- *Complete the sentence: A random variable is ...* *a function of the outcome of a random experiment.*

- *Give an example of two random variables X and Y that are uncorrelated but not independent.* *Let* (X, Y) *be equally likely to be any of the four pairs* $\{(-1, 0), (0, -1), (0, 1), (1, 0)\}$.

- *Prove that* $\text{var}(aX) = a^2\text{var}(X)$. *One has*

  $$\text{var}(aX) = E((aX)^2) - E(aX)^2 = a^2E(X^2) - a^2E(X)^2 = a^2[E(X^2) - E(X)^2] = a^2\text{var}(X).$$

- *Assume that X is exponentially distributed with rate 1, so that its probability density function is $f_X(x) = \exp\{-x\}$. Calculate $E(|X - 1|)$.*

  $$\begin{aligned}
  E(|X-1|) &= \int_0^1 (1-x)e^{-x}dx + \int_1^\infty (x-1)e^{-x}dx \\
  &= \int_0^1 e^{-x}dx - \int_0^1 xe^{-x}dx + \int_1^\infty xe^{-x}dx - \int_1^\infty e^{-x}dx \\
  &= \int_0^1 e^{-x}dx + \int_0^1 xde^{-x} - \int_1^\infty xde^{-x} - \int_1^\infty e^{-x}dx \\
  &= \int_0^1 e^{-x}dx + [xe^{-x}]_0^1 - \int_0^1 e^{-x}dx - [xe^{-x}]_1^\infty + \int_1^\infty e^{-x}dx - \int_1^\infty e^{-x}dx \\
  &= [xe^{-x}]_0^1 - [xe^{-x}]_1^\infty = 2e^{-1}.
  \end{aligned}$$

Problem 2. *(20%) The EE126 students fall into three categories. 10% of the students work extremely hard and have a 40% chance of getting an A in the class. 20% work very hard and have a 25% chance of getting an A. The remaining 70% work hard and have a 10% chance of getting an A.*

(a) What fraction of the students get an A?

(b) Given that a student did not get an A, is it more likely that he worked extremely hard, very hard, or just hard? What is the probability of that most likely event given that he did not get an A?

(c) Assume that you value an A in EE126 at $10,000.00 more than not getting an A because of the improved chance of getting into a great graduate school. Assume also that working extremely hard means 300 hours of work, working very hard means 200 hours, and working hard means 100 hours. Say that you value your time at $10.00 per hour. If you want to maximize your expected payoff minus the value of your time, should you work extremely hard, very hard, or only hard?

Let 1 mean EH, 2 mean VH, and 3 mean H. Let also $p_i = P(i)$ and $q_i = P[A|i] = 1 - P[A^c|i]$.

(a) We have

$$P(A) = \sum_{i=1}^{3} p_i q_i = 0.4 \times 0.1 + 0.25 \times 0.2 + 0.1 \times 0.7 = 0.16.$$

(b) The most likely cause of not getting an A is the value of i that maximizes $p_i(1 - q_i)$. In this case, it is $i = 3$, i.e., working only hard.

Also,

$$P[H \mid A^c] = \frac{p_3(1 - q_3)}{\sum p_i(1 - q_i)} = \frac{0.7 \times 0.9}{0.1 \times 0.6 + 0.2 \times 0.75 + 0.7 \times 0.9} = 0.75.$$

(c) Say that you work hard. With probability 0.1 you get an A, which is worth $10,000.00 to you, on average, but costs you 100.00 × $10.00 = $1,000.00. The net reward is then $10,000.00 × 0.1 - $1,000.00 = - $0.00. Similarly, if you work very hard, the net expected reward is $10,000× 0.25 -200 × $10.00 = $500.00. If you work extremely hard, the expected reward is $10,000 ×0.4 - 300 × $10.00 = $1,000.00. Thus, you should work extremely hard in EE126.

Problem 3. *(**20%**) You pick a point (X, Y) uniformly inside the unit circle $\{(x, y) \in \Re^2 \mid x^2 + y^2 \leq 1\}$. Calculate* var$(X)$*. [*Hint: You may want to consider $X^2 + Y^2$.*]*

By symmetry, $E(X) = 0$, so that $\text{var}(X) = E(X^2)$. Let $Z^2 = X^2 + Y^2$. We see that $P(Z \leq a) = a^2$ since Z is the distance from $(0, 0)$ to (X, Y), so that $Z \leq a$ if (X, Y) is picked in a circle with radius a. It follows that the p.d.f. of Z is $f_Z(z) = 2z$ for $z \in [0, 1]$.

Hence,

$$E(Z^2) = \int_0^1 z^2 2z dz = 1/2 = E(X^2 + Y^2) = 2E(X^2).$$

It follows that

$$\text{var}(X) = E(X^2) = 1/4.$$

Problem 4. *(15%) Show that*

$$var(X+Y) = var(X) + var(Y) + 2cov(X,Y).$$

One has

$$\begin{aligned}
\text{var}(X+Y) &= E((X+Y)^2) - [E(X+Y)]^2 \\
&= E(X^2 + 2XY + Y^2) - \{[E(X)]^2 + 2E(X)E(Y) + [E(Y)]^2\} \\
&= \{E(X^2) - [E(X)]^2\} + \{E(Y^2) - [E(Y)]^2\} + 2\{E(XY) - E(Y)E(Y)\} \\
&= \text{var}(X) + \text{var}(Y) + 2\text{cov}(X,Y).
\end{aligned}$$

Problem 5. *(**20%**) You choose two points X and Y independently and uniformly in $[0,1]$. You then look at the three segments that where created by cutting $[0,1]$ in the two places X and Y. What is the probability that the largest segment is less than 0.4?*

The three pieces have lengths $Z_1 = \min\{X, Y\}$, $Z_2 = \max\{X, Y\} - Z_1$, $Z_3 = 1 - Z_2$. The largest segment is less than 0.4 if

$$X \leq Y \text{ and } X \leq 0.4, 0.6 \leq Y \leq 0.4 + X$$

or

$$Y \leq X \text{ and } Y \leq 0.4, 0.6 \leq X \leq 0.4 + Y.$$

By symmetry, these two disjoint events have the same probability. To find the probability of the first event, we draw a picture.

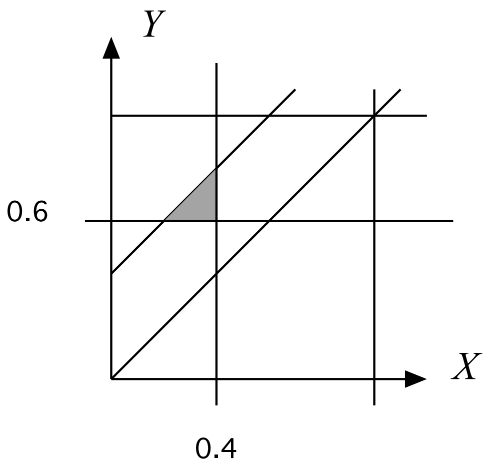

Figure 1. The first event.

We find that the probability of that event is $\frac{1}{2}0.2^2 = 0.02$. We conclude that the probability that the largest segment is less than 0.4 is

$$2 \times 0.02 = 0.04.$$