EE126: Probability and Random Processes **F'10**

Final — December 17

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Formulas: In the unlikely event you forgot them since the last midterm, here are a few potentially useful formulas:

$$\mathbf{X} = N(\mu, \Sigma) \Leftrightarrow f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\{-\frac{1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)\}$$

The LLSE of $\mathbf{X}$ given $\mathbf{Y}$ is $L[\mathbf{X}|\mathbf{Y}] = E(\mathbf{X}) + \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{Y}}^{-1}(\mathbf{Y} - E(\mathbf{Y}))$

$\text{cov}(\mathbf{AX}, \mathbf{BY}) = \mathbf{A}\text{cov}(\mathbf{X}, \mathbf{Y})\mathbf{B}^T$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $V = N(0,1)$, then $P(V > 1) = 0.159, P(V > 1.64) = 0.05,$
$P(V > 1.96) = 0.025, P(V > 2) = 0.023, P(V > 2.58) = 0.005.$

Problem 1. (**Short Problems 20%**)

- *Complete the sentence: A random variable is* *a measurable function of the outcome of a random experiment.*

- *Let X, Y be i.i.d. N(0, 1). Using the characteristics function, show that $Z = X + Y$ is Gaussian.*

  One has

$$\begin{aligned} E(\exp\{iuZ\}) &= E(\exp\{iu(X+Y)\}) = E(\exp\{iuX\})E(\exp\{iuY\}) \\ &= \exp\{-\frac{1}{2}u^2\}\exp\{-\frac{1}{2}u^2\} = \exp\{-\frac{1}{2}u^2\sigma^2\} \end{aligned}$$

  with $\sigma^2 = 2$, which shows that $Z = N(0,2)$.

- *Let X, Y be i.i.d. Exp(1). Derive the p.d.f. of $Z = X + Y$.*
  Clearly, $f_Z(z) = 0$ for $z < 0$. For $z \geq 0$, one has

$$\begin{aligned} f_Z(z) &= \int_0^\infty f_X(x)f_Y(z-x)dx = \int_0^z \exp\{-x\}\exp\{-(z-x)\}dx \\ &= \int_0^z \exp\{-z\}dx = z\exp\{-z\}. \end{aligned}$$

- *Let X be a random variable that takes values in $\{0, 1, 2, 3, \ldots\}$ and is such that*

$$P[X \geq n + m \mid X \geq n] = P(X \geq m), \forall m, n \geq 0.$$

What are the possible p.m.f.'s of X?

We find, for $n \geq 1$,

$$P(X \geq n+1) = P(X \geq n)P[X \geq n+1 \mid X \geq n] = P(X \geq n)P(X \geq 1) = P(X \geq n)q$$

with $q := P(X \geq 1)$. Thus,

$$P(X \geq n) = q^n, n \geq 1.$$

It follows that

$$P(X = n) = P(X \geq n) - P(X \geq n + 1) = q^n - q^{n+1} = q^n(1 - q) = (1 - p)^n p$$

with $p = 1 - q = P(X = 0)$. Thus, $X + 1 = G(p)$. That is, X is the number of coin flips before the first 'head.'

- *Let X, Y be i.i.d. U[-1, 1]. What is the LLSE $L[X^2|X + Y]$?*
Since $X^2 \perp X + Y$, one has $L[X^2|X + Y] = E(X^2) = 1/3$.

- *Give an example of a null recurrent irreducible discrete time Markov chain.*
The Markov chain with $P(n, n+1) = P(n, n-1) = 1/2$ for $n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

Problem 2. *(15%) Assume that, given $X = x$, $Y = N(2x + 3, 5)$. Assume also that $X = Exp(1)$.*

(a) What is $MLE[X \mid Y]$?

(b) What is $MAP[X \mid Y]$?

(a) $MLE[X \mid Y = y]$ is the value of x such that $2x + 3 = y$, i.e.,

$$MLE[X \mid Y] = \frac{Y}{2} - \frac{3}{2}.$$

(b) $MAP[X \mid Y = y] = \arg\max_x f_X(x) f_{Y|X}[y|x]$. Thus,

$$\begin{aligned} MAP[X \mid Y = y] &= \arg\max_x \exp\{-x\} \exp\{-(y - 2x - 3)^2/10\} \\ &= \arg\min_x g(x) \text{ with } g(x) := x + (y - 2x - 3)^2/10. \end{aligned}$$

Setting $dg(x)/dx = 0$, we find

$$1 - 4(y - 2x - 3)/10 = 0, \text{ i.e., } x = \frac{y}{2} - \frac{11}{4}.$$

Hence,

$$MAP[X \mid Y] = \frac{Y}{2} - \frac{11}{4}.$$

Problem 3. *(20%)*

The random vector (X, Y) is picked uniformly in the quarter circle

$$C := \{(x, y) \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}.$$

a) Sketch C and draw a guess for the LLSE $L[X|Y]$. Explain your guess in a few words. (The second part asks for a calculation, but a guess should help you check that result.)

Here is a sketch:

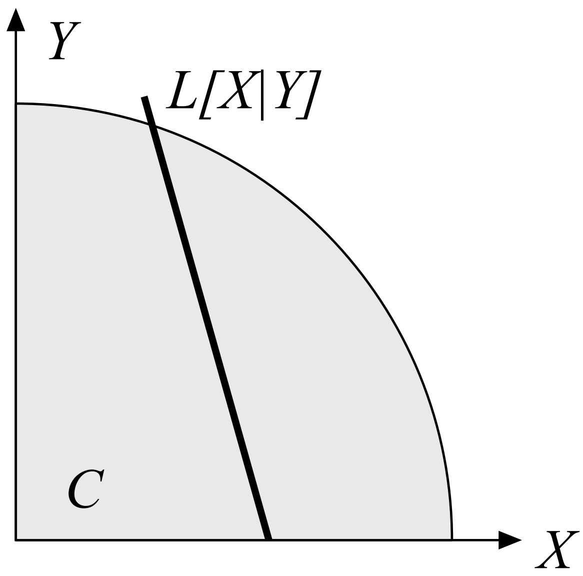

Figure 1. Sketch of C and $L[X|Y]$.

b) Calculate $L[X \mid Y]$.

We need to calculate $\Sigma_{XY} = E(XY) - E(X)E(Y)$ and $\Sigma_Y = \text{var}(Y) = E(Y^2) - E(Y)^2$. Observe that the density of (X,Y) is $\frac{4}{\pi}$ in C. Hence,

$$E(h(X,Y)) = \frac{4}{\pi}\int\int_C h(x,y)dxdy = \frac{4}{\pi}\int_0^1 [\int_0^{\sqrt{1-x^2}} h(x,y)dy]dx.$$

Thus,

$$\begin{aligned} E(X) = E(Y) &= \frac{4}{\pi}\int_0^1 [\int_0^{\sqrt{1-x^2}} ydy]dx \\ &= \frac{4}{\pi}\int_0^1 \frac{1}{2}(1-x^2)dx = \frac{2}{\pi}[1-\frac{1}{3}] = \frac{4}{3\pi}. \end{aligned}$$

Also,

$$\begin{aligned} E(XY) &= \frac{4}{\pi}\int_0^1 x[\int_0^{\sqrt{1-x^2}} ydy]dx \\ &= \frac{4}{\pi}\int_0^1 \frac{1}{2}(x-x^3)dx = \frac{2}{\pi}[\frac{1}{2}-\frac{1}{4}] = \frac{1}{2\pi}. \end{aligned}$$

Finally,

$$E(X^2+Y^2) = \frac{4}{\pi}\int\int_C (x^2+y^2)dxdy = \frac{4}{\pi}\int_0^{\pi/2}\int_0^1 r^2 rdrd\theta \text{ (since } dxdy = rdrd\theta)$$
$$= \frac{4}{\pi}\frac{\pi}{2}\int_0^1 r^3dr = \frac{4}{\pi}\frac{\pi}{2}\frac{1}{4} = \frac{1}{2},$$

so that $E(Y^2) = \frac{1}{4}$.

Combining these partial results, we find

$$\Sigma_{XY} = E(XY) - E(X)E(Y) = \frac{1}{2\pi} - (\frac{4}{3\pi})^2.$$

Also,

$$\text{var}(Y) = \frac{1}{4} - (\frac{4}{3\pi})^2.$$

Hence,

$$\begin{aligned} L[X|Y] &= E(X) + \Sigma_{XY}\Sigma_Y^{-1}(Y - E(Y)) \\ &= \frac{4}{3\pi} + [\frac{1}{2\pi} - (\frac{4}{3\pi})^2][\frac{1}{4} - (\frac{4}{3\pi})^2]^{-1}[Y - \frac{4}{3\pi}] \\ &\approx 0.55 - 0.3Y. \end{aligned}$$

Problem 4. *(15%) A monkey types random keys on a keyboard. Assume that each key that he types is equally likely to be any one of the 26 letters or a space. (We assume that there are no other symbols on the keyboard.) The monkey types 100 keys per second. In this problem, we explore the average time until the monkey types the symbols of "I WANT A BANANA" consecutively.*

a) Consider the following approximation. With some probability α, the monkey succeeds in 15 key strokes. With probability $1-\alpha$, the monkey has wasted some time T and has to try again, from scratch. What is then the average time A required with this approximation? What is α and what is a sensible value for T?

b) Formulate the problem as the first hitting time of a Markov chain. [Hint: Be careful of what happens when the monkey types "I".]

c) Write the first step equations for the mean value of the hitting time of the Markov chain, with the appropriate boundary conditions. (Do not solve.)

a) The average time A should be such that $A = \alpha(0.15) + (1-\alpha)(T + A)$, so that

$$A = 0.15 + \frac{1-\alpha}{\alpha}T \approx \frac{T}{\alpha}.$$

One has $\alpha = (1/27)^{15}$ and $T \approx (27/26)10^{-2}$ since T is the average time until failure, and every attempt fails with probability $26/27$. Thus, one expects

$$A \approx (27)^{15}10^{-2} \approx 3.10^{19}s \approx \text{one trillion years}.$$

b) Let X_n be the number of correct successive symbols that the monkey produces after n keystrokes. Then X_n is a Markov chain with

$$P(0,1) = p := 1/27, P(0,0) = 1-p;$$
$$P(k,k+1) = P(k,1) = p := 1/27 \text{ and } P(k,0) = q := 1-2p, k = 1,\ldots,14.$$

Indeed, if the monkey has typed the first k symbols of the sentence, he has one chance in 27 of hitting the correct key next. If he does not, if he hits "I", he has typed one correct consecutive symbol and otherwise, he has hit 0 correct symbol.

c) The first step equations are as follows:

Let $\beta(k) = E[T_{15}|X_0 = k]$ where $T_{15} = \min\{n \geq 0 \mid X_n = 15\}$. Then

$$\beta(0) = 1 + p\beta(1) + (1-p)\beta(0);$$
$$\beta(k) = 1 + p\beta(k+1) + p\beta(1) + q\beta(0), k = 1,\ldots,14$$
$$\beta(15) = 0.$$

Problem 5. *(15%)*

You are given three light bulbs with i.i.d. exponentially distributed life times with mean 1 year.

a) You turn one light bulb on until it burns out, then turn on the second one. What is the probability that the two bulbs burn out before one year?

b) You turn the three light bulbs on at the same time. What is the probability that at least two bulbs burn out before one year? [Hint: Let V be the time when the first bulb burns out and Y be the time when the second one burns out. What is the p.d.f. of V? Using the memoryless property of the exponential distribution, you can obtain easily the distribution of $Y - V$.]

a) Let X_1, X_2, X_3 be the life times of the bulbs. We have

$$P(X_1 + X_2 \leq 1) = \int_0^1 f_{X_1}(x)P(X_2 \leq 1 - x)dx = \int_0^1 e^{-x}[1 - e^{-(1-x)}]dx$$
$$= \int_0^1 e^{-x}dx - \int_0^1 e^{-1}dx = 1 - 2e^{-1} \approx 0.26$$

b) The time V when the first bulb burns out is exponentially distributed with rate 3. By the memoryless property, the time Y when the second bulb burns out is such that $W = Y - V$ is independent of V and is exponentially distributed with rate 2. Hence,

$$P(Y \leq 1) = P(V + W \leq 1) = \int_0^1 f_V(x)P(W \leq 1 - x)dx$$
$$= \int_0^1 3e^{-3x}[1 - e^{-2(1-x)}]dx$$
$$= 3\int_0^1 e^{-3x}dx - 3\int_0^1 e^{-2-x}dx$$
$$= [1 - e^{-3}] - 3e^{-2}[1 - e^{-1}] \approx 0.69.$$

Problem 6. *(15%) A target has location* $\mathbf{X} = N(0, \Sigma)$ *in* $\Re^2$*. Your radar indicates that the target is at location* $\mathbf{Y} = \mathbf{X} + \mathbf{Z}$ *where* $\mathbf{Z} = N(0, \sigma^2 \mathbf{I})$ *and is independent of* $\mathbf{X}$*. You want to detonate a bomb that destroys the target if it explodes within a radius* r *of the target.*

a) Where should you place the bomb to maximize the probability of destroying the target?

b) Particularize the result to the case where

$$\Sigma = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \text{ and } \sigma^2 = \frac{1}{4}.$$

a) We want to choose the location $\mathbf{a}$ that maximizes $P(||\mathbf{X} - \mathbf{a}||^2 \leq r^2)$. By symmetry, the best location is $\mathbf{a} = \mathbf{X}^* := E[\mathbf{X} \mid \mathbf{Y}]$.

Now,

$$E[\mathbf{X} \mid \mathbf{Y}] = \Sigma_{XY} \Sigma_Y^{-1} \mathbf{Y} = \Sigma[\Sigma + \sigma^2 \mathbf{I}]^{-1} \mathbf{Y}.$$

b) We have

$$E[\mathbf{X} \mid \mathbf{Y}] = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 5/4 & 2 \\ 2 & 21/4 \end{bmatrix}^{-1} \mathbf{Y} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \frac{16}{41} \begin{bmatrix} 21/4 & -2 \\ -2 & 5/4 \end{bmatrix} \mathbf{Y} = \frac{4}{41} \begin{bmatrix} 5 & 2 \\ 2 & 9 \end{bmatrix} \mathbf{Y}$$