Fall 2009: EECS126 Final

No Collaboration Permitted. Two sheets of notes permitted. Turn in with your exam.

Be clear and precise in your answers

Write your name and student ID number on every sheet.

Come to the front if you have a question.

110 points is a very good score. There are more points than that on the exam. So look over the entire exam. Some points at the end may be easier to get than others earlier in the exam.

Problem 1.1 *(36pts) True or False. Prove or show a counterexample:*

a. *12pts. Let A_1, A_2, A_3 be events with $0 < P(A_3) < 1$. Suppose A_1 and A_2 are conditionally independent given A_3 and are also conditionally independent given A_3^c. Then A_1 and A_2 must be independent.*

b. 12pts. For any random variable X and any $a > 0$ we have

$$P(|X| < a) \leq a^2 E[\frac{1}{X^2}]$$

c. 12pts. *X and Y are two random variables with some joint distribution and each one individually has a well defined positive noninfinite variance. Then the mean-squared estimation error random variable $\tilde{X} = (X - E[X|Y])$ is independent of Y.*

Problem 1.2 (45pts) *Markov Fun (the parts of this problem are not connected to each other, so do them in any order)*

a. *25pts. You have a server that can be in one of two states: "busy" or "idle." Assume that the states evolve in discrete-time in a Markov fashion. Nature dictates that the probability of going from "idle" to "busy" is $\frac{9}{10}$. However, we have control of the probability q of going from "busy" to "idle."*

   - **Draw this Markov chain, labeling all the transition probabilities.**
   - **Calculate the stationary distribution as a function of q.**
   - **How low can you bring the steady state probability of the server being busy?**
   - **Assume that the system starts in the stationary distribution. Conditioned on the state being "busy" at time 3 what is the probability of it being "idle" at time 2?**

Extra paper in case you need it.

b. *4pts. Draw a Markov chain (and label all the transition probabilities) whose probability transition matrix has eigenvalues +1 and 0.*

c. *4pts. Draw a Markov chain (and label all the transition probabilities) whose probability transition matrix has eigenvalues +1 and -1.*

d. *4pts. Draw a Markov chain (and label all the transition probabilities) whose probability transition matrix has eigenvalues +1 and +1.*

e. 4pts. Draw a Markov chain (and label all the transition probabilities) whose probability transition matrix has eigenvalues +1 and $-\frac{1}{5}$.

f. 4pts. Draw a Markov chain (and label all the transition probabilities) with two distinct recurrent classes and at least one transient state.

Problem 1.3 (15 pts.) *There is some unknown probability p of a microchip being bad, and we know that different chips go bad independently of each other.* **How many chips should we sample in order to be able to estimate p to within ± 0.05? (Justify your answer in some detail and feel free to leave any unevaluatable integrals unevaluated)**

Problem 1.4 (15 pts.) *Somebody claims that it is sufficient to have just* 2 *bathrooms on an airplane that carries* 100 *people on it, each of which wants to go to the bathroom independently of the others with some probability* q*. You probe this person about what does he mean by "sufficient" and he says that he means that less than 10% of the time will someone have to wait for an empty bathroom assuming that time is slotted and the time required to finish using the bathroom is* 1 *slot long.*

What can you conclude about q? Give your analysis and explain how you are modeling the problem using probability.

Extra page in case you need more room

Problem 1.5 (15 pts.) *Consider a networking system in which packets arrive according to a Poisson process with rate λ arrivals per second on average. I tell you that exactly* 1 *arrival has occurred between time* 0 *and some time* t*.* **Conditioned on this information, what is the distribution of the exact arrival time for this arrival?**

Show as complete a derivation as you can.

Problem 1.6 (25 pts.) *X and Y are jointly Gaussian random variables. We know that $E[X] = E[Y] = 0$ and $E[X^2] = 1$ and $E[Y^2] = 2$. Furthermore, $E[XY] = -\frac{1}{2}$. Suppose that I tell you that X, Y are both obtainable as linear combinations of independent standard (zero mean and unit variance) Gaussian random variables W, V.* **Do you have enough information to uniquely write X and Y as linear combinations of W, V? If so, prove it and give these unique linear combinations. If not, give at least two distinct linear combinations that are both compatible with the specified information.**