EE 126 Fall 2007 Midterm #1
Thursday October 4, :30–5pm

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

- You have 90 minutes to complete the quiz.
- Write your solutions in the exam booklet. We will not consider any work not in the exam booklet.
- This quiz has three problems that are in no particular order of difficulty.
- You may give an answer in the form of an arithmetic expression (sums, products, ratios, factorials) of <u>numbers</u> that could be evaluated using a calculator. Expressions like $\binom{8}{3}$ or $\sum_{k=0}^{5}(1/2)^k$ are also fine.
- A correct answer does not guarantee full credit and a wrong answer does not guarantee loss of credit. You should concisely indicate your reasoning and show all relevant work. The grade on each problem is based on our judgment of your level of understanding as reflected by what you have written.
- This is a closed-book exam except for one single-sided, handwritten, 8.5×11 formula sheet plus a calculator.
- Be neat! If we can't read it, we can't grade it.
- At the end of the quiz, turn in your solutions along with this quiz (this piece of paper).

Problem		Score
1	[13 points]	
2	[15 points]	
3	[13 points]	
Total 41		

Problem 1: (13 points)

Each computer chip produced at a factory has a probability d of being defective (D), independent of other chips.

(a) 2 pts What is the probability of finding k defective chips in a sample of n chips?

(b) 3 pts Now suppose that chips are tested in sequence, one by one, until a total of k defective chips are discovered. Find the probability that the k^{th} defective chip is found after exactly n chips are sampled.

Now suppose that the testing procedure is unreliable:

$$\textbf{Missed Detection:} \quad \mathbb{P}[\text{test says OK} \mid \text{chip D}] = t_{MD}$$
$$\textbf{False Alarm:} \quad \mathbb{P}[\text{test says D} \mid \text{chip OK}] = t_{FA}$$

for some $t_{MD}, t_{FA} \in (0,1)$.

(c) 4 pts For this part, assume that $t_{MD} = t_{FA} = t$. Suppose that each chip is tested 10 times independently, and rejected if at least 6 of the tests come up defective. Given that a chip is rejected, compute the probability that it is defective as a function of t and d.

(d) 4 pts For this part, assume that $t_{FA} = 0$, but $t_{MD} > 0$. Suppose that each chip is tested once, and rejected if the test declares it defective. Given that the $(n+1)$ chip is the first one to be rejected, let Z be the number of chips in sequence before it that were defective (but not discovered by testing). Compute the PMF of Z. (It should be a function of d and t_{MD}.)

Solution:

(a) This is the binomial distribution. $\binom{n}{k} d^k (1-d)^{n-k}$.

(b) The last chip has to be defective: otherwise, we place $(k-1)$ defective chips in $n-1$ positions, so that the probability is $\binom{n-1}{k-1} d^k (1-d)^{n-k}$.

(c) We use Bayes rule. Let A be the event that 6 or more of the tests come up defective. We have

$$\begin{aligned}
\mathbb{P}[A \mid D] &= \sum_{\ell=6}^{10} \binom{10}{\ell} (1-t)^{\ell} t^{10-\ell} \\
\mathbb{P}[A] &= d \sum_{\ell=6}^{10} \binom{10}{\ell} (1-t)^{\ell} t^{10-\ell} + (1-d) \sum_{\ell=6}^{10} \binom{10}{\ell} (t)^{\ell} (1-t)^{10-\ell}
\end{aligned}$$

Hence, by Bayes rule, we have

$$\begin{aligned}
\mathbb{P}[D \mid A] &= \frac{\mathbb{P}[A \mid D]\mathbb{P}[D]}{\mathbb{P}[A]} \\
&= \frac{d \sum_{\ell=6}^{10} \binom{10}{\ell} (1-t)^{\ell} t^{10-\ell}}{d \sum_{\ell=6}^{10} \binom{10}{\ell} (1-t)^{\ell} t^{10-\ell} + (1-d) \sum_{\ell=6}^{10} \binom{10}{\ell} (t)^{\ell} (1-t)^{10-\ell}}.
\end{aligned}$$

(d) Let Y be the event that all defective chips are missed. The probability that all k defective chips out of n chips are missed is $P(Z=k,Y)=\binom{n}{k}(t\cdot d)^k(1-d)^{n-k}$

$$P(Z=k|Y)=\frac{P(Z=k,Y)}{\sum_{\ell=0}^{n}P(Z=\ell,Y)}=\frac{\binom{n}{k}(t\cdot d)^k(1-d)^{n-k}}{\sum_{\ell=0}^{n}\binom{n}{\ell}(t\cdot d)^\ell(1-d)^{n-\ell}}$$

We can reduce the expression by collapsing the binomial expansion in the denominator:

$$\begin{aligned}
&\frac{\binom{n}{k}(t\cdot d)^k(1-d)^{n-k}}{\sum_{\ell=0}^{n}\binom{n}{\ell}(t\cdot d)^\ell(1-d)^{n-\ell}}\\
=&\frac{\binom{n}{k}(t\cdot d)^k(1-d)^{n-k}}{(t\cdot d+1-d)^n}\\
=&\frac{\binom{n}{k}(t\cdot d)^k(1-d)^{n-k}}{(t\cdot d+1-d)^z(t\cdot d+1-d)^{n-z}}\\
=&\binom{n}{k}(\frac{t\cdot d}{t\cdot d+1-d})^k(\frac{1-d}{t\cdot d+1-d})^{n-k}
\end{aligned}$$

The final expression can be found more directly by thinking of the conditional situation as its own experiment. We know that each of the first n chips is one of two types: undefective or defective and missed. In the original experiment we had three types, but in the new conditional frame the defective and caught type is impossible, and therefore we must re-noramlize the probability. We compute:

$$\begin{aligned}
&\mathbb{P}(\{\text{chip is defective and missed}\}\mid\{\text{chip is defective and missed OR chip is undefective}\})\\
=&\frac{\mathbb{P}(\{\text{chip is defective and missed}\})}{\mathbb{P}(\{\text{chip is defective and missed}\})+\mathbb{P}(\{\text{chip is undefective}\})}\\
=&\frac{t\cdot d}{t\cdot d+1-d}
\end{aligned}$$

With the above parameter calculated, we simply construct a binomial distribution with it as the success paramter and n as the number of trials.

Problem 2: (15 points)

Lucky Bob walks into the casino with k dollars in his pocket, and starts to play roulette. On each spin of the roulette wheel, he wins 1 dollar with probability $q < \frac{1}{2}$, and loses a dollar with probability $1-q$, with each spin being independent of every other.

(a) 2 pt Let V_i be a discrete random variable with $\mathbb{P}[V_i = 1] = q$ and $\mathbb{P}[V_i = -1] = 1-q$. Compute $\mathbb{E}[V_i]$.

(b) 5 pts Suppose that Bob gambles for k rounds and then stops: let X be the amount of money left in his pocket.

(i) Compute the expectation and variance of X.

(ii) Compute the PMF of X given that he leaves the casino with 2 dollars or less.

(c) 3 pts Suppose that Bob continues gambling until either he runs out of money (0 in his pocket), or until he has T dollars in his pocket (assume $T > k$). Let p_k be the probability that he leaves the casino with no money. Prove that $p_k = q\, p_{k+1} + (1-q)p_{k-1}$ for all $0 < k < T$, and $p_0 = 1$ and $p_T = 0$.

(d) 5 pts Now suppose that $k \geq 5$, and that Bob leaves the casino after he loses for the fifth time. Let Z be the amount of money in his pocket when leaves the casino.

(i) Compute $\mathbb{E}[Z]$.

(ii) Compute the expectation of Z given that he does not lose in the first 9 rounds.

Solution:

(a) We have $\mathbb{E}[V_i] = 1 \cdot q + (-1) \cdot (1-q) = 2q - 1$.

(b) (i) We write $X = k + V_1 + V_2 + \ldots + V_k$ where each V_i is $+1$ with probability q, and -1 with probability $(1-q)$, as above.

We have $\mathbb{E}[X] = \mathbb{E}\left[k + \sum_{i=1}^k V_i\right] = k + \sum_{i=1}^k 2q - 1 = 2qk$.

Also, $\mathrm{var}(X) = \sum_{i=1}^k \mathrm{var}(V_i)$ by independence of the gambling rounds, and so $\mathrm{var}(X) = k\left(1 - (2q-1)^2\right)$ since $V_i^2 = 1$ always.

(ii) It is only possible for X to take on one of two values: 0 or 2. Therefore, we simply need to calculate $\mathbb{P}\left(\{X=0\} \mid \{X=0\} \cup \{X=2\}\right)$ and $\mathbb{P}\left(\{X=2\} \mid \{X=0\} \cup \{X=2\}\right)$.

$$
\begin{aligned}
\mathbb{P}(\{X=0\} \mid \{X=0\} \cup \{X=2\} &= \frac{\mathbb{P}(\{X=0\})}{\mathbb{P}(\{X=0\}) + \mathbb{P}(\{X=2\})} \\
&= \frac{(1-q)^k}{(1-q)^k + \binom{k}{1}(1-q)^{k-1}q^1} \\
&= \frac{1-q}{1+(k-1)q} \\
&= a
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}(\{X=2\} \mid \{X=0\} \cup \{X=2\} &= 1 - \mathbb{P}(\{X=0\} \mid \{X=0\} \cup \{X=2\} \\
&= \frac{kq}{1+(k-1)q} \\
&= b
\end{aligned}
$$

Where a and b have been defined for convenience. The PMF of X conditioned on our conditioning event C s given by:

$$
p_{X|C}(x) = \begin{cases} a & x = 0 \\ b & x = 2 \\ 0 & \text{otherwise} \end{cases}
$$

(c) We can show the relation using total probability.

With total probability, we know the following must be true:

$$
\begin{aligned}
& \mathbb{P}(\{\text{leaves with 0 and started with k}\}) \\
= \; & \mathbb{P}(\{\text{wins first}\})\mathbb{P}(\{\text{leaves casino with 0 and started with k}\} \mid \{\text{wins first}\}) \\
& + \mathbb{P}(\{\text{loses first}\})\mathbb{P}(\{\text{leaves casino with 0 and started with k}\} \mid \{\text{loses first}\})
\end{aligned}
$$

The conditioning is equivalent to starting the process over with a different starting value of k, and so we have:

$$
p_k = qp_{k+1} + (1-q)p_{k-1}
$$

(d) Let T_i be geometric random variables with parameter $(1-q)$. We can write $Z = k + \sum_{i=1}^{5}(T_i - 2)$ so that

$$
\mathbb{E}[Z] = k + 5\ (\mathbb{E}[T_i] - 2) = (k-10) + \frac{5}{1-q}.
$$

Given that he does not lose in the first 9 rounds, we can think as the experiment as being the same, except that he now has $k+9$ dollars in his pocket to start. Hence this conditional expectation is equal to $k - 1 + \frac{5}{1-q}$.

Problem 3: (13 points)

A chess board consists of 64 squares, arranged in an 8×8 array (see Figure 1(i)). Squares can be occupied by zero or one pieces.

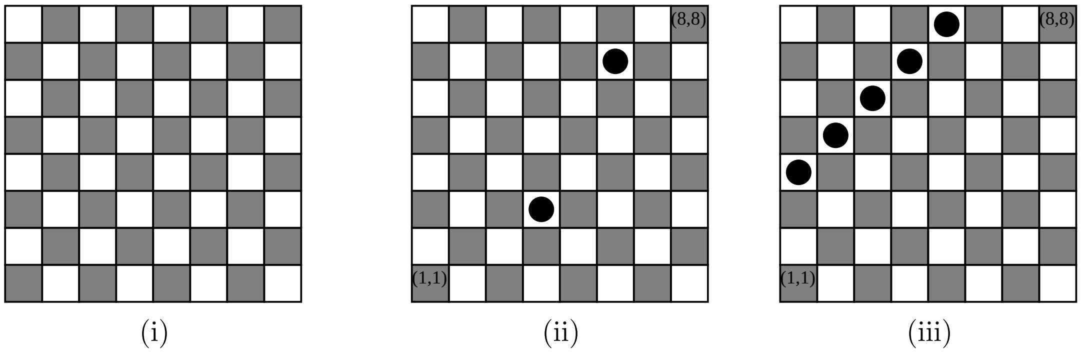

Figure 1: (i) 8×8 chess board. (ii) Chess board with two positions blocked. (iii) Chess board with five positions blocked.

For parts (a) through (c), consider the *unblocked* chess board, in Figure 1(i).

(a) 2 pts How many ways are there to place 8 pawn pieces on a chess board?

(b) 3 pts How many ways are there to place 8 pawn pieces so that no pair shares the same row or column?

(c) 4 pts How many ways are there to place 6 pawn pieces and 2 rook pieces so that the rooks are in adjacent columns?

For parts (d) and (e), a rook can move either one step to the right or one step upwards.

(d) 3 pts Suppose that two positions on the chessboard are occupied (see Figure 1(ii)). How many different paths can the rook take from $(1,1)$ to $(8,8)$ without going through either of the two occupied positions?

(e) 1 pt Now suppose that five positions are blocked, as shown in Figure 1(iii). How many different paths from $(1,1)$ to $(8,8)$ does the rook now have?

Solution:

(a) Pawns are indistinguishable, and so the intent of the problem was to not count the order of the pawns, but instead only the number of board configurations. However, since "number of ways to place" might be interpreted as instead counting the sequences of placement, both counting order and not counting order were acceptable answers. However, points were deducted for inconsistent interpretations.

To count without order, this problem requires us to divide the board squares into two groups: the group of squares with pawns on them, and the group of squares without pawns on them. The two groups are of known size, and so the number can be computed directly with a binomial coefficient: $\binom{64}{8}$.

(b) We certainly need one pawn in each column, and so we can decompose the problem via the counting principle into deciding where to place the pawn for each column.

In the first column, we have 8 options for which square to place a pawn, corresponding to the 8 rows. In the second, we have 7, since the placement of the pawn in the first column has eliminated one of the potential rows. Notice that at this point in the counting tree, no matter which row was chosen for the first pawn, we always have 7 options for this second pawn, and so the counting principle will work here. We continue the process for the rest of the rows and take the product of the options we have at each decision junction to get the answer 8!.

The above was a solution for un-ordered pawns, since we have effectively chosen 8 spots for the 8 pawns satisfying the criteria without specifying the order in which they are placed (careful not to confuse the application of the counting principle with ordering the pawns themselves; to make the order discrepancy clear, think of the pawns as each having one of 8 different colors). To count the order of the pawns, we must multiply by another factor of 8!, which is the number of permutations of a set of size 8, and so we arrive at $(8!)^2$ as the number of sequences in which to place the 8 pawns.

(c) We will decompose the problem into the following sequence of decisions: first we choose which pair of columns the rooks will be in (there are 7 options, one for each placement of the "left" of the two columns)), then we place the rooks (one must be in each column, giving us $8 \cdot 8 = 64$ options), and then we place the pawns in the remaining spots on the board, in a procedure analogous to that of part (a), giving us $\binom{62}{6}$ options. The product across these choices is $7 \cdot 64 \cdot \binom{62}{6}$.

To apply order to the above decisions, we can order the rooks by adding a factor of 2 and order the pawns by a factor of 6!.

(d) Our basic working unit is the ability to count the number of unrestricted paths from one square to another using a similar formula to the one in the homework: we think of moving from $(1, 1)$ to some (m, n) as a sequence of $(m - 1 + n - 1)$ total moves, with $m - 1$ upwards moves and $n - 1$ moves to the right in any order. We can count such sequences as $\binom{m+n-2}{n-1}$, and therefore the number of unrestricted paths is given by the same expression.

We can also count the paths through a particular square by decomposing the path into the "before" path and the "after" path and applying the counting principle. For example, the number of paths from $(1, 1)$ to $(8, 8)$ through $(4, 3)$ can be calculated by first finding the number of paths to $(4, 3)$ from $(1, 1)$ using our formula, which yields $\binom{5}{2}$. We can then find the number of paths from $(4, 3)$ to $(8, 8)$ as $\binom{9}{4}$. We can choose a path through $(4, 3)$ by first choosing the path up to $(4, 3)$ and then choosing the path from $(4, 3)$ to $(8, 8)$, so we form the product of the two expressions to get the number of paths through $(4, 3)$.

To count the paths that do not pass through the blocked spots, we compute the following:

$$\begin{aligned}
\text{\# paths not through blocked} &= \text{\# of unrestricted paths} \\
&\quad -\text{\# of paths through (4,3) block} \\
&\quad -\text{\# of paths through (6,7) block} \\
&\quad +\text{\# of paths through both blocks}
\end{aligned}$$

We need the final term because we have double-subtracted the number of paths through both blocks: the paths through $(4,3)$ include those also through $(6,7)$ and vice-versa, so therefore we have subtracted them from the total twice and must add them back in. This is generally termed the inclusion-exclusion formula, where $|A+B| = |A|+|B|-|A \cap B|$.

Substituting numerical values into the above decomposition, we have the answer:

$$\binom{14}{7} - \binom{5}{2}\binom{9}{4} - \binom{11}{5}\binom{3}{1} + \binom{5}{2}\binom{6}{2}\binom{3}{1}$$

(e) We could try a technique like the one above, but with so many blocks, the expression would quickly turn nasty. Instead, we can explot the geometry of the blocking positions by using the reflection principle.

We again change the problem to counting the "complement": to find the number of paths avoiding the blocks, we take the total number of paths and subtract the number of paths that must hit the blocked squares. To compute the latter, we reflect the portion of the board directly beneath the blocked squares across the blocking line, so that it creates an extension of the board to the left. We then see that the number of paths from the lower-left of the reflected portion to the position $(8,8)$ must have the same number as the paths from $(1,1)$ to $(8,8)$ that touch the blocked positions, since those positions from the reflected start to $(8,8)$ must cross the blocked positions. By the reflection principle we have a bijection, and so we count the unrestricted paths from the reflection to $(8,8)$ for the following numerical answer:

$$\binom{14}{7} - \binom{14}{4}$$