EE 126 Fall 2007 Midterm #1
Thursday October 4, 3:30–5pm

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

- You have 90 minutes to complete the quiz.
- Write your solutions in the exam booklet. We will not consider any work not in the exam booklet.
- This quiz has three problems that are in no particular order of difficulty.
- A correct answer does not guarantee full credit and a wrong answer does not guarantee loss of credit. You should concisely indicate your reasoning and show all relevant work. The grade on each problem is based on our judgment of your level of understanding as reflected by what you have written.
- This is a closed-book exam except for one single-sided, handwritten, 8.5×11 formula sheet plus a calculator.
- Be neat! If we can't read it, we can't grade it.
- At the end of the quiz, turn in your solutions along with this quiz (this piece of paper).

Problem		Score
1	[13 points]	
2	[15 points]	
3	[13 points]	
Total		

Problem 1: (13 points)

Each computer chip produced at a factory has a probability d of being defective (D), independent of other chips.

(a) 2 pts What is the probability of finding k defective chips in a sample of n chips?

(b) 3 pts Now suppose that chips are tested in sequence, one by one, until a total of k defective chips are discovered. Find the probability that the k^{th} defective chip is found after exactly n chips are sampled.

Now suppose that the testing procedure is unreliable:

$$\textbf{Missed Detection:} \quad \mathbb{P}[\text{test says OK} \mid \text{chip D}] = t_{MD}$$
$$\textbf{False Alarm:} \quad \mathbb{P}[\text{test says D} \mid \text{chip OK}] = t_{FA}$$

for some $t_{MD}, t_{FA} \in (0,1)$.

(c) 4 pts For this part, assume that $t_{MD} = t_{FA} = t$. Suppose that each chip is tested 10 times independently, and rejected if at least 6 of the tests come up defective. Given that a chip is rejected, compute the probability that it is defective as a function of t and d.

(d) 4 pts For this part, assume that $t_{FA} = 0$, but $t_{MD} = t > 0$. Suppose that each chip is tested once, and rejected if the test declares it defective. Given that the $(n+1)$ chip is the first one to be rejected, let Z be the number of chips in sequence before it that were defective, but not discovered by testing. Compute the PMF of Z. (It should be a function of n, d and t.)

Problem 2: (15 points)

Lucky Bob walks into the casino with k dollars in his pocket, and starts to play roulette. On each spin of the roulette wheel, he wins 1 dollar with probability $q < \frac{1}{2}$, and loses a dollar with probability $1 - q$, with each spin being independent of every other.

(a) 2 pts Let V_i be a discrete random variable with $\mathbb{P}[V_i = 1] = q$ and $\mathbb{P}[V_i = -1] = 1 - q$. Compute $\mathbb{E}[V_i]$.

(b) 5 pts Suppose that Bob gambles for k rounds and then stops: let X be the amount of money left in his pocket.

- (i) Compute the expectation and variance of X.
- (ii) Compute the PMF of X given that he leaves the casino with 2 dollars or less. (Assume $k > 2$).

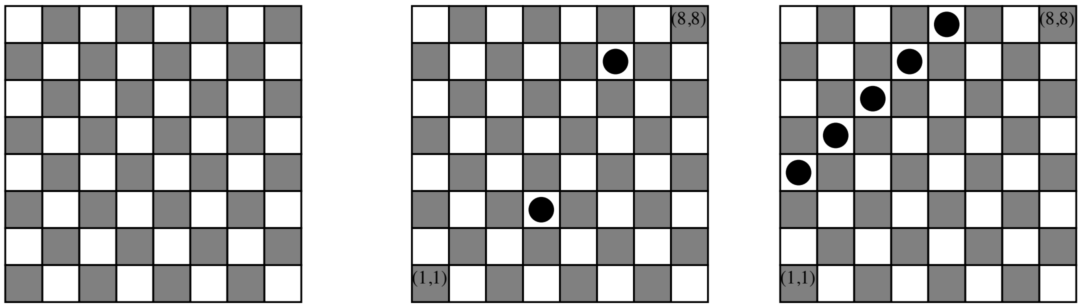
(c) 3 pts Suppose that Bob continues gambling until either he runs out of money (0 in his pocket), or until he has T dollars in his pocket (assume $T > k$). Let p_k be the probability that he leaves the casino with no money if he starts with k dollars. Show that $p_k = q\, p_{k+1} + (1 - q)p_{k-1}$ for all $0 < k < T$.

(d) 5 pts Now suppose that $k \geq 5$, and that Bob leaves the casino after he loses for the fifth time. Let Z be the amount of money in his pocket when leaves the casino.

- (i) Compute $\mathbb{E}[Z]$.
- (ii) Compute the expectation of Z given that he does not lose in the first 9 rounds.

Problem 3: (13 points)

A chess board consists of 64 squares, arranged in an 8×8 array (see Figure 1(i)). Squares can be occupied by zero or one pieces.



(i) (ii) (iii)

Figure 1: (i) 8×8 chess board. (ii) Chess board with two positions blocked. (iii) Chess board with five positions blocked.

For parts (a) through (c), consider the *unblocked* chess board, in Figure 1(i).

(a) 2 pts How many ways are there to place 8 pawn pieces on a chess board?

(b) 3 pts How many ways are there to place 8 pawn pieces so that no pair shares the same row or column?

(c) 4 pts How many ways are there to place 6 pawn pieces and 2 rook pieces so that the rooks are in adjacent columns?

For parts (d) and (e), a rook can move either one step to the right or one step upwards.

(d) 3 pts Suppose that two positions on the chessboard are occupied (see Figure 1(ii)). How many different paths can the rook take from $(1, 1)$ to $(8, 8)$ without going through either of the two occupied positions?

(e) 1 pt Now suppose that five positions are blocked, as shown in Figure 1(iii). How many different paths from $(1, 1)$ to $(8, 8)$ does the rook now have?