EE 126 Fall 2006 Final Exam
Tuesday, December 19: 12:30–3:30pm

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

- You have 3 hours to complete the exam.
- Write your solutions in the exam booklet. We will not consider any work not in the exam booklet.
- This quiz has five (5) problems that are in no particular order of difficulty.
- A correct answer does not guarantee full credit and a wrong answer does not guarantee loss of credit. You should concisely indicate your reasoning and show all relevant work. The grade on each problem is based on our judgment of your level of understanding as reflected by what you have written.
- This is a closed-book exam except for two handwritten, 8.5×11 formula sheets plus a calculator.
- Be neat! If we can't read it, we can't grade it.

Some useful formulae

(a) **Geometric sums:** For $|a| < 1$, $\sum_{k=0}^{\infty} a^k = 1/(1-a)$.

(b) For any $k = 1, 2, 3, \ldots$, $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

(c) For any γ and $0 \le a < b < +\infty$, we have

$$\int_a^b x \exp(\gamma x) dx \quad = \quad \frac{be^b - ae^a}{\gamma} - \frac{1}{\gamma^2}\left[e^b - e^a\right].$$

(d) **Gaussian normalization:** For any $\mu \in \mathbb{R}$, we have

$$\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}(x-\mu)^2\right) dx \quad = \quad \sqrt{2\pi}.$$

(e) **Binomial formula:** For any $a, b \in \mathbb{R}$ and positive integer n, we have

$$(a+b)^n \quad = \quad \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Problem 1: (20 points) Consider a pair of continuous random variables (X, Y) with joint PDF uniform over the shaded region in Figure 1:

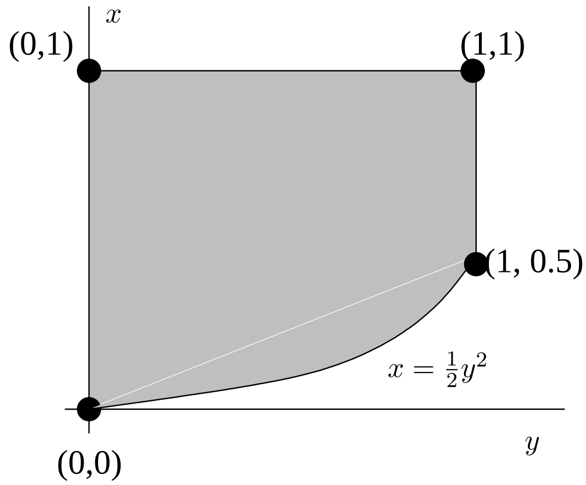

Figure 1: Joint PDF of random variables X and Y.

(a) (4 pt) Compute the conditional PDF $p_{X|Y}(x \mid y)$.

(b) (5 pt) Compute the linear least-squares estimator (LLSE) of X based on Y.

(c) (5 pt) Compute the minimum mean-squared error estimator (MMSE) of X based on Y.

(d) (6 pt) Now suppose that in addition to observing Y, you are also told that $X \geq \frac{1}{2}Y$. Compute the LLSE and MMSE estimators of X based on both Y and $\{X \geq \frac{1}{2}Y\}$.

Problem 2: Rolling dice: (20 points)

You play a game by rolling six fair dice simultaneously. Each die has sides $\{1, 2, 3, 4, 5, 6\}$, and each of your rolls are independent. You win a pet frog if at least 2 of the dice shows the same number, and you win a pet elephant if there are at least 4 sixes. What is the probability of

(a) (4 pt) rolling exactly 2 sixes?

(b) (4 pt) winning a pet frog?

(c) (4 pt) winning a pet elephant?

(d) (3 pt) winning a pet elephant given that you have won a pet frog?

(e) (5 pt) winning a pet elephant given that the total score (on all your six rolls) is greater than or equal to 33?

Problem 3: Loaded elevators: (20 points)

An elevator is designed to tolerate a maximum weight of at most 5000 pounds. We are interested in the probability that the elevator is *overloaded*, meaning that the total weight of all people onboard exceeds the threshold of 5000 pounds. Assume the weight W_i of person i can be modeled as an exponential variable with parameter $\lambda = 1/150$, so that $f_{W_i}(w) = \lambda \exp(-\lambda w)$ for $w \geq 0$, and that the weights of different people are independent.

First assume that exactly 26 people climb onboard the elevator.

(a) (4 pt) Using Markov's inequality, compute an upper bound on the probability that the elevator is overloaded.

(b) (5 pt) Using the the central limit theorem, compute an approximation to the probability that the elevator is overloaded. (You may specify your answer in terms of the CDF $\Phi(z) = \mathbb{P}(Z \leq z)$ of a standard normal variable $Z \sim N(0, 1)$.)

Now suppose that a *random number* T of people board the elevator, where T is a geometric random variable with parameter $q \in (0, 1)$.

(c) (5 pt) Using Chebyshev's inequality, compute an upper bound on the probability that the elevator is overloaded, as a function of q.

(d) (6 pt) Now suppose that you observe that at least 20 people have boarded the elevator (so you know that $T \geq 20$). Compute the minimum mean-squared estimator (MMSE) of the total weight of people on the elevator given this information.

Problem 4: Homer Simpson at the power plant: (20 points)

One day, as Homer Simpson is working at the nuclear power plant, one of the reactors begins to melt down. The emission of radioactive particles can be modeled as a Poisson process with rate $\lambda = 100$ particles/second.

(a) (2 pt) What is the expected *total* number of particles that escape the time window $[10, 20] \cup [50, 60]$ seconds?

(b) (3 pt) Suppose that exactly 300 particles escaped in the first second. What is the PDF of the number of particles that escape in the next second?

In order to stop the spread of radiation, the plant is equipped with a set of n shields that are either "ON" or "OFF". Any OFF shield has no effect on the particle stream, whereas any ON shield blocks each particle with probability p, and let it through with probability $1 - p$, independently of all other particles. Suppose that each shield acts independently of all the other shields.

(c) (4 pt) For some fixed integer k (with $1 \leq k \leq n$), suppose that exactly k of the shields are ON, and consider the stochastic process defined by the particles that end up escaping. Prove that the expected number of particles that escape per second is equal to $100\,(1-p)^k$. Is this a Poisson process?

(d) (7 pt) Now suppose that Homer is distracted by eating donuts, so he only turns ON each shield (independently of all other shields) with probability 0.50.

Let X_n represent the number of particles that escape in the first second. (Recall that fixed integer n is the total number of shields.)

(i) Show that $\mathbb{E}[X_n] = 100\,(1 - \frac{p}{2})^n$.

(ii) Compute $\text{var}[X_n]$.

(*Hint:* Using conditional expectation, the result of (c) could be helpful, and a formula from p. 2 could be useful to you.)

(e) (4 pt) Does the sequence $\{X_1, X_2, X_3 \ldots\}$ converge in probability to any real number c? If not, explain why not. If so, give the value of c, and justify the convergence in probability.

Problem 5: (20 points) Every day that he leaves work at 2pm, Albert the Absent-minded Professor toggles his light switch according to the following protocol: (i) if the light is on, he switches it off with probability 0.80; and (ii) if the light is off, he switches it on with probability 0.30. At no other time (other than the end of each day) is the light switch touched.

(a) (2 pt) Suppose that on Monday night, Albert's office is equally likely to be light or dark. What is the probability that his office will be lit all five nights of the week (Monday through Friday) at 11pm each night?

(b) (3 pt) Suppose that you observe that his office is lit on both Monday and Friday nights. Compute the expected number of nights, from that Monday through Friday inclusive (5 possible nights total), that his office is lit.

(c) (3 pt) Suppose that Albert's office is lit on Monday night. Compute the expected number of days until the first night that his office is dark.

Now suppose that Albert has been working for five years.

(d) (3 pt) Is his light more likely to be on or off at the end of a given workday?

(e) (4 pt) What is the probability of his office light staying off for five consecutive nights?

Now suppose that for each night i the office light is left on, the electricity cost is exponentially distributed with parameter $\lambda = 1$ dollars, whereas if the light is left off, then the electricity cost is 0 dollars. Let Z_i represent the electricity cost on night i, and let $Z = \sum_{i=1}^{365} Z_i$ be the total electricity cost over one year.

(f) (5 pt) What is the expected total electricity cost for Albert's office light over the course of one year? Can you use the central limit theorem presented in class to approximate the probability $\mathbb{P}[Z \geq 365]$? If so, compute such an approximation; if not, explain why not.