Department of EECS - University of California at Berkeley
EECS126 - Probability and Random Processes - Fall 2003
Midterm No. 2: 11/14/2003

SOLUTIONS

There are five questions, worth 20% each. Answer on these sheets. Show your work. Good luck.

Question 1. Let $\{X, Y, Z\}$ be independent $N(0, 1)$ random variables.
a.(14%) Calculate

$$E[3X + 5Y \mid 2X - Y, X + Z].$$

b. (6%) How does the expression change if X, Y, Z are i.i.d. $N(1, 1)$?

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a. Let $V_1 = 2X - Y, V_2 = X + Z$ and $\mathbf{V} = [V_1, V_2]^T$. Then

$$E[3X + 5Y \mid \mathbf{V}] = \mathbf{a}\Sigma_V^{-1}\mathbf{V}$$

where

$$\mathbf{a} = E((3X + 5Y)\mathbf{V}^T) = [1, 3]$$

and

$$\Sigma_V = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}.$$

Hence,

$$E[3X + 5Y \mid \mathbf{V}] = [1, 3]\begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}^{-1}\mathbf{V} = [1, 3]\frac{1}{6}\begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}\mathbf{V}$$
$$= \frac{1}{6}[-4, 13]\mathbf{V} = -\frac{2}{3}(2X - Y) + \frac{13}{6}(X + Z).$$

b. Now,

$$E[3X + 5Y \mid \mathbf{V}] = E(3X + 5Y) + \mathbf{a}\Sigma_V^{-1}(\mathbf{V} - E(\mathbf{V})) = 8 + \frac{1}{6}[-4, 13](\mathbf{V} - [1, 2]^T)$$
$$= \frac{26}{6} - \frac{2}{3}(2X - Y) + \frac{13}{6}(X + Z).$$

Question 2. 25%. Let X, Y be independent random variables uniformly distributed in $[0,1]$. Calculate $L[Y^2 \mid 2X+Y]$.

One has

$$L[Y^2 \mid 2X+Y] = E(X^2) + \frac{E(Y^2(2X+Y)) - E(Y^2)E(2X+Y)}{\text{var}(2X+Y)}(2X+Y-E(2X+Y))$$

$$= \frac{1}{3} + \frac{1/3+1/4-(1/3)(3/2)}{4(1/3-1/4)+(1/3-1/4)}(2X+Y-3/2).$$

Question 3. 15%. Let $\{X_n, n \geq 1\}$ be independent $N(0,1)$ random variables. Define $Y_{n+1} = aY_n + (1-a)X_{n+1}$ for $n \geq 0$ where Y_0 is a $N(0, \sigma^2)$ random variable independent of $\{X_n, n \geq 0\}$. Calculate

$$E[Y_{n+m}|Y_0, Y_1, \ldots, Y_n]$$

for $m, n \geq 0$.

Hint: First argue that observing $\{Y_0, Y_1, \ldots, Y_n\}$ is the same as observing $\{Y_0, X_1, \ldots, X_n\}$. Second, get an expression for Y_{n+m} in terms of $Y_0, X_1, \ldots, X_{n+m}$. Finally, use the independence of the basic random variables.

One has

$$\begin{aligned} Y_{n+1} &= aY_n + (1-a)X_{n+1}; \\ Y_{n+2} &= aY_{n+1} + (1-a)X_{n+2} = a^2 Y_n + (1-a)X_{n+2} + (1-a)^2 X_{n+1}; \\ &\cdots \\ Y_{n+m} &= a^m Y_n + (1-a)X_{n+m} + (1-a)^2 X_{n+m-1} + \cdots + (1-a)^m X_{n+1}. \end{aligned}$$

Hence,

$$E[Y_{n+m} \mid Y_0, Y_1, \ldots, Y_n] = a^m Y_n.$$

Question 4. 20%. Given θ, the random variables $\{X_n, n \geq 1\}$ are i.i.d. $U[0, \theta]$. Assume that θ is exponentially distributed with rate λ.

a. Find the MAP $\hat{\theta}_n$ of θ given $\{X_1, \ldots, X_n\}$.

b. Calculate $E(|\theta - \hat{\theta}_n|)$.

One finds that

$$f[x \mid \theta]f(\theta) = \frac{1}{\theta^n} 1\{x_k \leq \theta, k = 1, \ldots, n\}\lambda e^{-\lambda\theta}.$$

Hence,

$$\hat{\theta}_n = max\{X_1, \ldots, X_n\}.$$

Consequently, by symmetry,

$$E[\theta - \hat{\theta}_n|\theta] = \frac{1}{n+1}\theta.$$

Finally,

$$E(|\theta - \hat{\theta}_n|) = E(E[\theta - \hat{\theta}_n|\theta])) = \frac{1}{\lambda(n+1)}.$$

A few words about the symmetry argument. Consider a circle with a circumference length equal to 1. Place $n + 1$ point independently and uniformly on that circumference. By symmetry, the average distance between two points is $1/(n+1)$. Pick any one point and open the circle at that point, calling one end 0 and the other end 1. The other n points are distributed independently and uniformly on $[0, 1]$. So, the average distance between 1 and the closest point in $1/(n + 1)$. Of course, we could do a direct calculation.

Question 5. 20%. Let (X, Y) be jointly Gaussian. Show that $X - E[X \mid Y]$ is Gaussian and calculate its mean and variance.

We know that

$$E[X \mid Y] = E(X) + \frac{cov(X, Y)}{\mathrm{var}(Y)}(Y - E(Y)).$$

Consequently,

$$X - E[X \mid Y] = X - E(X) - \frac{cov(X, Y)}{\mathrm{var}(Y)}(Y - E(Y))$$

and is certainly Gaussian. This difference is zero-mean. Its variance is

$$\mathrm{var}(X) + [\frac{cov(X, Y)}{\mathrm{var}(Y)}]^2 \mathrm{var}(Y) - 2\frac{cov(X, Y)}{\mathrm{var}(Y)} cov(X, Y) = \mathrm{var}(X) - \frac{[cov(X, Y)]^2}{\mathrm{var}(Y)}.$$