EE 121 Digital Communication Systems
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University of California, Berkeley: Spring 2005
April 14, 2005

Second Midterm Exam

Last name	First name	SID

- You have two hours to complete this exam.
- There are 100 points for this exam. Points for the individual problems and subproblems are marked in the problem statement.
- The exam is closed-book and closed-notes; calculators, computing and communication devices are *not* permitted.
- However, one handwritten and *not photocopied* double-sided sheet of notes is allowed.
- No form of collaboration between the students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.
- If we can't read it, we can't grade it.
- We can only give partial credit if you write out your derivations and reasoning in detail.
- You may use the back of the pages of the exam if you need more space.

*** GOOD LUCK! ***

Problem	Points earned	out of
Problem 1		40
Problem 2		25
Problem 3		20
Problem 4		15
Total		100

Useful formulas.

In this class (thanks a lot to the textbook!), the Fourier transform is defined as

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt, \tag{1}$$

which makes the inverse Fourier transform

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df. \tag{2}$$

The Fourier transform of

$$y(t) = \text{sinc}(t/T) \tag{3}$$

is given by

$$Y(f) = \begin{cases} T, & |f| \leq \frac{1}{2T} \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Problem 1 *(Signaling over the additive white Gaussian noise (AWGN) channel.)* 40 Points

It is intended to use the following M signals for transmission over the AWGN channel. For $m = 1$,

$$s_1(t) = \begin{cases} 1, & 0 \leq t < \frac{2T}{M} \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

and for $m = 2, 3, \ldots, M$,

$$s_m(t) = \begin{cases} 1, & \frac{(m-2)T}{M} \leq t < \frac{(m-1)T}{M} \\ -1, & \frac{(m-1)T}{M} \leq t \leq \frac{mT}{M} \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

(a) (2 Pts) For $M = 4$, sketch the signals.

(b) (5 Pts) Show that the following functions are an orthonormal basis for the signal set $s_m(t)$ given in Eqn. (5) and (6).

$$\psi_m(t) = \begin{cases} \sqrt{M/T}, & \frac{(m-1)T}{M} \leq t < \frac{mT}{M} \\ 0, & \text{otherwise,} \end{cases} \tag{7}$$

for $m = 1, \ldots, M$. *Remark:* Recall from Midterm 1 which questions need to be addressed. Briefly outline the answer to each of these questions.

(c) (6 Pts) For $M = 2$, sketch the signal constellation. For $M = 4$, write out the four signal vectors and calculate the distances between them.

(d) (6 Pts) For general M, give the signal constellation points in vector notation.

(e) (6 Pts) For $M = 5$, sketch the correlation-type demodulator. Carefully specify what each box is doing. If you plot functions, label the axes. Denote the received signal by $r(t)$.

(f) (5 Pts) Can the correlation-type demodulator be implemented by a single filter, followed by a sampler? If so, give a system diagram. Carefully specify what each box is doing.

(g) (10 Pts) Now suppose that this signal set is used for transmission over the standard additive white Gaussian noise channel as studied in class, with noise of power spectral density $N_0/2$. For general M, give an upper bound on the error probability for the case when all M signals are used with equal probability. Carefully justify each step in your derivation.

Problem 2 *(Bandlimited AWGN channels.)* 25 Points

Consider transmission across the bandlimited AWGN channel that we studied in class. As we discussed in class, the transmitted signal can be expressed as

$$v(t) = \sum_{i=-\infty}^{\infty} a_i g(t - iT), \tag{8}$$

where $g(t)$ is an appropriately selected pulse, and the corresponding received signal is

$$\begin{aligned} r(t) &= \int_{\tau=-\infty}^{\infty} c(t-\tau)v(\tau)d\tau + n(t) && (9) \\ &= \sum_{i=-\infty}^{\infty} a_i (c * g)(t - iT) + n(t), && (10) \end{aligned}$$

where the channel impulse response $c(t)$ has Fourier transform $C(f)$ given by

$$C(f) = \begin{cases} 1, & \text{if } |f| \le W, \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

and $n(t)$ is additive white Gaussian noise of power spectral density $N_0/2$.

The signal $r(t)$ is then passed through the matched filter $m_g(t)$ for the pulse $g(t)$ to obtain

$$\begin{aligned} y(t) &= \sum_{i=-\infty}^{\infty} a_i (m_g * c * g)(t - iT) + (m_g * n)(t) && (12) \\ &= \sum_{i=-\infty}^{\infty} a_i x(t - iT) + (m_g * n)(t). && (13) \end{aligned}$$

(a) (7 Pts) Bonnie suggest using the pulse $g(t) = \frac{1}{\sqrt{T}}\text{sinc}(t/T)$, together with *antipodal* modulation, that is, $a_i = \pm\sqrt{\mathcal{E}_a}$.

- Select the symbol interval T (as a function of W) such that intersymbol interference is avoided. A brief justification, for example in the shape of a sketch, is acceptable.
- Determine $x(t = 0)$. Find the power spectral density of the filtered noise $(m_g * n)(t)$ and determine the variance of each noise sample.
- Write the formula for the error probability as a function of $\mathcal{E}_a$ and N_0. A short justification is acceptable.

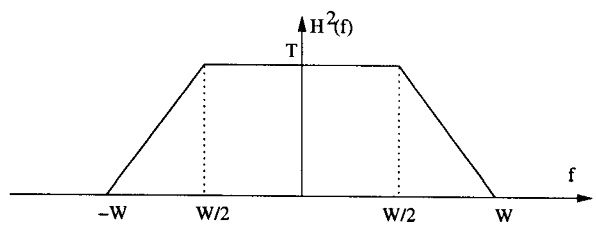


Figure 1: The square of the spectrum of the pulse used in Problem 2, Part (b).

(b) (10 Pts) Her colleague, Clyde suggests using the pulse $h(t)$ whose spectrum is shown in Figure 1, (the spectrum $H(f)$ is real-valued, and the figure shows its *square* $H^2(f)$). The coefficients a_i are picked from 4-ary PAM, i.e., $a_i \in \{-3c, -c, c, 3c\}$, where c is related to the signal energy.

- Select the symbol interval T (as a function of W) such that intersymbol interference is avoided. A brief justification, for example in the shape of a sketch, is acceptable.
- Determine $x(t=0)$. Find the power spectral density of the filtered noise $(m_h * n)(t)$ and determine the variance of each noise sample.
- Write the formula for the error probability as a function of c and N_0. A short justification is acceptable.

(c) (8 Pts) Compare the systems of Bonnie and Clyde. You can compare rates, error probabilities, transmitted power, implementation issues, and any other feature that you can think of. *Hint:* You can start with fixing the transmit power to be P, i.e., the energy for each time slot of duration T to be $\mathcal{E}_T = PT$.

Be sure to justify any claims you make with sufficient evidence and explanations. We are looking for your ability to identify, analyze, and justify tradeoffs between these two system designs.

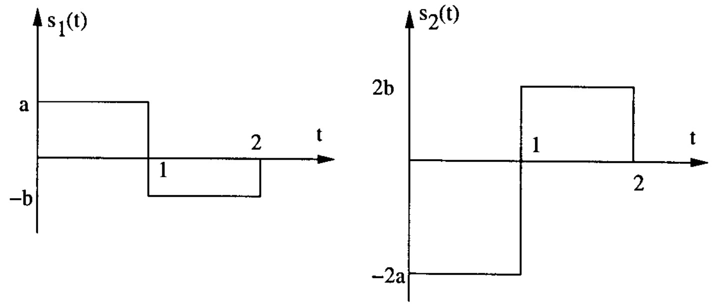


Figure 2: The signals for Problem 3. Take $a = 2/\sqrt{5}$ and $b = 1/\sqrt{5}$.

Problem 3 *(A special AWGN channel.)* 20 Points

Consider the signal set illustrated in Figure 2. Transmission is over an AWGN channel of a special kind: If you transmit $s_1(t)$, then the received signal is

$$r(t) \quad = \quad s_1(t) + n(t), \tag{14}$$

where $n(t)$ is additive white Gaussian noise of power spectral density $1/2$. However, if you transmit $s_2(t)$, the received signal is

$$r(t) \quad = \quad s_2(t) + \tilde{n}(t), \tag{15}$$

where $\tilde{n}(t)$ is additive white Gaussian noise of power spectral density 2.

(a) (1 Pt) Label the axis and the two signal points in Figure 3.

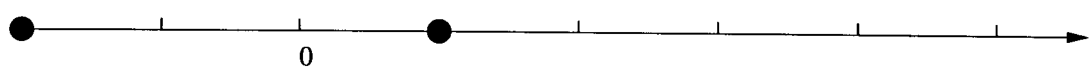


Figure 3: Signal space for Problem 2. Label the axis.

(b) (6 Pts) As we have seen in class, the correlation-type demodulator will first determine

$$r \stackrel{\text{def}}{=} \int_0^2 r(t) s_1(t) dt. \tag{16}$$

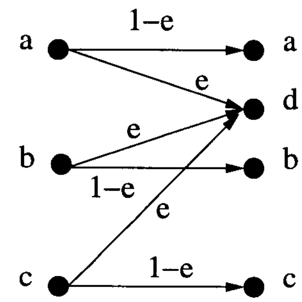
Find the conditional distributions $p(r|s_1)$ and $p(r|s_2)$.

(c) (7 Pts) Determine the maximum likelihood (ML) decision rule to decide, based on the received signal r, whether s_1 or s_2 was transmitted. Sketch the rule geometrically in Figure 3. Use $\ln(2) \approx 3/4$.

(d) (6 Pts) Determine the resulting error probability.

Problem 4 *(Channel Capacity.)* 15 Points

Consider the channel whose conditional probability mass function is sketched in the figure on the left. Clearly, $e \leq 1$.



(a) (5 Pts) Calculate $H(X|Y)$ in terms of $H(X)$ and e. *Hint:* Consider separately the cases $H(X|Y = a), H(X|Y = b), H(X|Y = c)$, and $H(X|Y = d)$.

(b) (4 Pts) Using your result from Part *(a)*, write $I(X;Y)$ and find the capacity C of this channel.

(c) (2 Pts) How many bits can one push through this channel on average, by means of a very sophisticated encoder and decoder of block length n, in such a way that the error probability drops to zero as n goes to infinity?

Answer: .... Bits per ....

Note: No justification is needed here. If you did not solve Part *(b)*, assume that the answer was $C = 1$.

(d) (4 Pts) A rover on Mars is looking for traces of hematite to determine if there was water there at one point in time. Suppose that the probability of finding hematite is independent from hour to hour (because of the distance between points that the rover visits) and that the probability of finding hematite is p, while the probability of finding no hematite is $1 - p$. The rover would like to communicate a long sequence of these observations back to a satellite without making any errors (in the limit as the length of the block of observations goes to infinity). To communicate, the rover can use the channel studied in this problem once per hour (because of power limitations). Sketch a block diagram of a system that achieves this. For what values of p and e does your system work? *Remark:* Take $\log_2 3 = 1.6$.