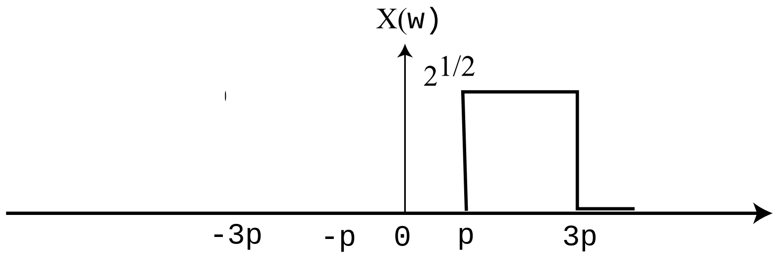

EECS 120. Midterm No. 2, March 24, 2000. Solution.

1. **20 points**

   (a) Plot the Fourier Transform $X(\omega)$ of a signal $x \in ContSignals$ whose total energy is 2 and such that $X(\omega) = 0$ for $|\omega - 2\pi| > \pi$.

   (b) Now find the time-domain signal x by taking the inverse FT of X.

   Answer Take X as shown in Figure 1. By Parseval's relation its energy is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \frac{1}{2\pi} \times 2\pi \times 2 = 2.$$

   Recall that

$$y(t) = \frac{\sin(\pi t)}{\pi t} \overset{FT}{\longleftrightarrow} Y(\omega) = 1, |\omega| \leq \pi, \ = 0, |\omega| > \pi. \tag{1}$$

   Observe that $X(\omega) = 2^{1/2}Y(\omega - 2\pi)$, and so

$$x(t) = 2^{1/2}y(t)e^{j2\pi t}.$$

2. **15 points** Fill in the blanks.

(a) The LT of $x(t) = tu(t)$ is $\frac{1}{s^2}$ and its ROC is $Real(s) > 0$.

(b) The LT of $x(t) = e^{-t}u(t)$ is $\frac{1}{s+1}$ and its FT is $\frac{1}{j\omega+1}$.

(c) The transfer function $H(s) = \frac{s-1}{s+1}$ of an LTI system has a pole at ______________ and its impulse response is $h(t) =$ ______________.

Answer (c) $H(s) = \frac{s-1}{s+1}$ has a pole at $s = -1$. Using partial fraction expansion, $H(s) = 1 - \frac{2}{s+1}$. Taking inverse LT gives

$$h(t) = \delta(t) - 2e^{-t}u(t).$$

3. **20 points** Find the solution $y(t), t \geq 0$, of the differential equation

$$\ddot{y}(t) - 3\dot{y}(t) + 2y(t) = 0,$$

with initial condition $y(0-) = 1, \dot{y}(0-) = 1$. Check that your solution satisfies these initial conditions.

Answer Taking LT of the differential equation gives,

$$s^2Y(s) - sy(0-) - \dot{y}(0-) - 3sY(s) + 3y(0-) + 2Y(s) = 0,$$

so, substituting for initial conditions,

$$Y(s) = \frac{s-2}{s^2 - 3s + 2} = \frac{1}{s-1} .$$

Taking inverse LT gives

$$y(t) = e^t u(t).$$

Check initial conditions: $y(0) = 1$, and $\dot{y}(0) = 1$.

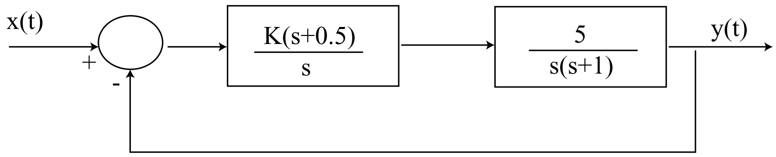

Figure 1: System for Problem 4

4. **20 points** In Figure 1 K is a real constant. Find the closed-loop transfer function $H(s)$. Use the Routh test to determine the values of K for which H is stable.

Answer The closed-loop transfer function is

$$H(s) = \frac{G}{1+G},$$

where $G(s) = \frac{K(s+0.5)}{s} \times \frac{5}{s(s+1)}$. Substituting gives, after some simplification,

$$H(s) = \frac{5K(s+0.5)}{s^3 + s^2 + 5Ks + 2.5K}.$$

The Routh test on the denominator $D(s) = s^3 + s^2 + 5Ks + 2.5K$ gives

$$\begin{array}{c|cc} s^3 & 1 & 5K \\ s^2 & 1 & 2.5K \\ s^1 & 2.5K & 0 \\ s^0 & 2.5K & 0 \end{array}$$

So H is stable if and only if $K > 0$.

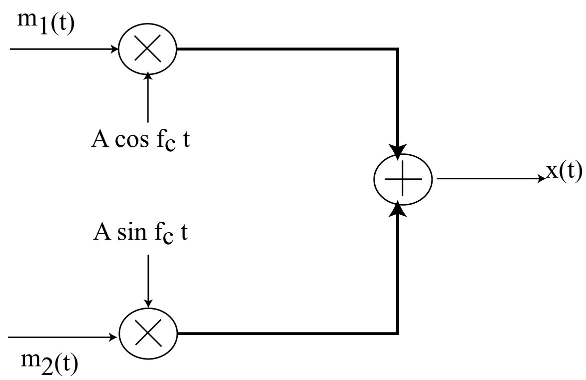

Figure 2: System for Problem 5

5. **25 points** In Figure 2 m_1 and m_2 are real signals with real Fourier Transforms $M_1(f)$ and $M_2(f)$ respectively. Suppose that $M_i(f) = 0$, for $|f| > 15$ kHz. The carrier frequency $f_c = 100$ kHz.

   (a) Determine the Fourier Transform $X(f)$ of the modulated signal x. Write an expression for $|X(f)|$. What is the bandwidth of x?

   (b) Find a scheme to demodulate x and recover both signals m_1 and m_2. Prove that your scheme works.

Answer We have

$$\begin{aligned} x(t) &= Am_1(t)\cos 2\pi f_c t + Am_2(t)\sin 2\pi f_c t \\ &= \frac{A}{2}m_1(t)[e^{j2\pi f_c t} + e^{-j2\pi f_c t}] + \frac{A}{2j}m_2(t)[e^{j2\pi f_c t} - e^{-j2\pi f_c t}] \\ \longleftrightarrow & \; X(f) = \frac{A}{2}[M_1(f-f_c) + M_1(f+f_c)] + \frac{A}{2j}[M_2(f-f_c) - M_2(f+f_c)] \end{aligned}$$

and since M_1, M_2 are real,

$$\begin{aligned} |X(f)| &= \frac{A}{2}\left\{[M_1(f-f_c) + M_1(f+f_c)]^2 + [M_2(f-f_c) + M_2(f+f_c)]^2\right\} \\ &= \frac{A}{2}\left\{[M_1^2(f-f_c) + M_2^2(f-f_c)]^{1/2} + [M_1^2(f+f_c) + M_2^2(f+f_c)]^{1/2}\right\}^{1/2}. \end{aligned}$$

The bandwidth of X is 30 kHz, twice the bandwidth of m_1 (and m_2).

To demodulate, we multiply x by $\cos 2\pi f_c t$ and pass the result through a LPF and separately multiply x by $\sin 2\pi f_c t$ and pass the result through a LPF. Thenb

$$\begin{aligned} x(t)\cos 2\pi f_c t &= Am_1(t)\cos^2(2\pi f_c t) + Am_2(t)\sin(2\pi f_c t)\cos(2\pi f_c t) \\ &= \frac{A}{2}m_1(t)[1 + 2\cos(4\pi f_c t)] + \frac{A}{2}m_2(t)\sin(4\pi f_c t). \end{aligned}$$

If this signal is passed through a LPF with bandwidth 15 KHz, the output signal will be $\frac{A}{2}m_1(t)$, since the other signals are located near $2f_c = 200$ KHz.

Similarly, $x(t)\sin 2\pi f_c t$, passed through a LPF will give as output the signal $\frac{A}{2}m_2(t)$.