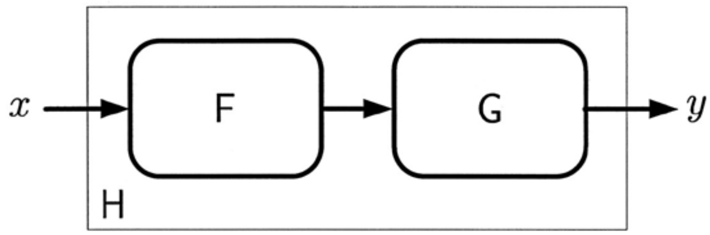
MT2.1 (35 Points) We form a system H by placing a pair of discrete-time LTI systems F and G in cascade, as shown below.



Let f, g, and h denote the impulse responses of the systems F, G, and H, respectively. Moreover, $\widehat{F}$, $\widehat{G}$, and $\widehat{H}$ are the transfer functions of the systems F, G, and H, respectively.

The impulse responses and transfer functions of the three systems are described below:

$$f(n) = \begin{cases} 1 & n = 0, 1, 2 \\ 0 & \text{elsewhere.} \end{cases} \qquad \longleftrightarrow \quad \widehat{F}(z) = 1 + z^{-1} + z^{-2}$$

$$g(n) = \begin{cases} +1 & n = 0, 3, 5, 8 \\ -1 & n = 1, 4, 7 \\ 0 & \text{elsewhere.} \end{cases} \qquad \longleftrightarrow \quad \widehat{G}(z) = 1 - z^{-1} + z^{-3} - z^{-4} + z^{-5} - z^{-7} + z^{-8}$$

$$h(n) = \begin{cases} 1 & n = 0, 5, 10 \\ 0 & \text{elsewhere.} \end{cases} \qquad \longleftrightarrow \quad \widehat{H}(z) = 1 + z^{-5} + z^{-10}.$$

(a) (10 Points) Determine the region of convergence of $\widehat{G}(z)$, the transfer function of the subsystem G.

Since $g(n)$ is a finite-length signal, the RoC of $\hat{G}(z)$ is the entire $\mathbb{C}$-plane except possibly $z = 0$ and/or ∞.

Since $g(n)$ is causal, $z = \infty$ is in the RoC.

$$\text{RoC}(G) = \{ z : 0 < |z| \}$$

(b) (20 Points) Show that the frequency response of the subsystem G can be expressed as

$$G(\omega) = A(\omega)\, e^{i\alpha\omega},$$

where $A(\omega)$ is a real-valued *amplitude* (not necessarily magnitude) function, and α is a constant. Determine a reasonably simple expression for $A(\omega)$ and the numerical value of α.

$$G(\omega) = \hat{G}(z)\Big|_{z=e^{i\omega}} = 1 - e^{-i\omega} + e^{-3i\omega} - e^{-4i\omega} + e^{-5i\omega} - e^{-7i\omega} + e^{-8i\omega} =$$

The middle term is $e^{-4i\omega}$. We have:

$$G(\omega) = e^{-4i\omega}\left(e^{4i\omega} - e^{3i\omega} + e^{i\omega} - 1 + e^{-i\omega} - e^{-3i\omega} + e^{-4i\omega}\right) =$$

$$e^{-4i\omega}\left(\underbrace{2\cos 4\omega - 2\cos 3\omega + 2\cos\omega - 1}_{\text{real}}\right)$$

Thus, we can substitute $A(\omega) = 2\cos 4\omega - 2\cos 3\omega + 2\cos\omega - 1$

$\alpha = -4$

(c) (25 Points) Provide a well-labeled pole-zero diagram for $\widehat{G}(z)$. To help you plot, you might want to know that $2\pi/15$ radians is approximately 24 degrees.

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Note $\hat{G}(z) = \dfrac{\hat{H}(z)}{\hat{F}(z)} = \dfrac{1+z^{-5}+z^{-10}}{1+z^{-1}+z^{-2}} = \dfrac{z^{10}+z^{5}+1}{z^{8}(z^2+z+1)}$

Poles where $z^8(z^2+z+1)=0$:

$z^8=0 \Rightarrow$ 8 poles @ $z=0$

$z^2+z+1=0 \Rightarrow$ poles @ $z = \dfrac{-1\pm\sqrt{1^2-4\cdot1\cdot1}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} = e^{\pm i\frac{2\pi}{3}}$ (*)

However, from part (a) and since there are no poles in the RoC, we know $e^{\pm i\frac{2\pi}{3}}$ will be cancelled by zeros.

Zeros where $z^{10}+z^5+1=0$: From (*)

$z^2+z+1 = (z-e^{i2\pi/3})(z-e^{-i2\pi/3})$

$\Rightarrow z^{10}+z^5+1 = (z^5-e^{i2\pi/3})(z^5-e^{-i2\pi/3}) = 0$

$$\left.\begin{array}{l} z^5 - e^{i2\pi/3} = 0 \Rightarrow z = e^{i(2\pi/15+\frac{2\pi}{5}k)} \\ z^5 - e^{-i2\pi/3} = 0 \Rightarrow z = e^{i(-2\pi/15+\frac{2\pi}{5}k)} \end{array}\right\} k=0,\dots,4$$

Zeros at $z=e^{i\theta}$ for

$\theta = \frac{2\pi}{15}, \frac{8\pi}{15}, \frac{14\pi}{15}, \boxed{\frac{20\pi}{15}}, \frac{26\pi}{15}$

$-\frac{2\pi}{15}, \frac{4\pi}{15}, \boxed{\frac{10\pi}{15}}, \frac{16\pi}{15}, \frac{22\pi}{15}$

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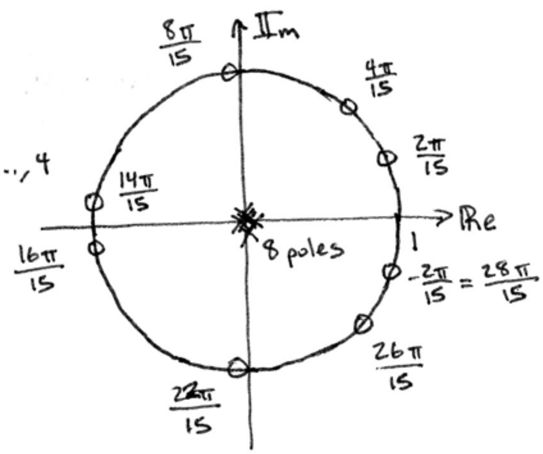


(d) (15 Points) If the input to the cascade interconnection H is the signal x described by

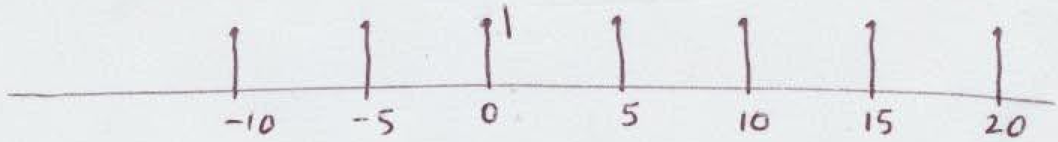
$$\forall n \in \mathbb{Z}, \quad x(n) = \sum_{\ell=-\infty}^{+\infty} \delta(n - 15\ell),$$

determine and provide a well-labeled plot of the corresponding output signal y.

$\hat{H}(z) = 1 + z^{-5} + z^{-10} \Rightarrow h(n) = \delta(n) + \delta(n-5) + \delta(n-10)$

$y(n) = x(n) * h(n) = x(n) * \delta(n) + x(n) * \delta(n-5) + x(n) * \delta(n-10) =$

$x(n) + x(n-5) + x(n-10) = \sum_{\ell=-\infty}^{+\infty} \delta(n-15\ell) + \delta(n-5-15\ell) + \delta(n-10-15\ell)$



(e) (15 Points) Express the frequency response $H(\omega)$ of the cascade interconnection H in terms of the frequency response $F(\omega)$ of the subsystem F.

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We observe that $\hat{H}(z) = \hat{F}(z^5)$

In the frequency domain, this corresponds to $H(\omega) = F(5\omega)$

Another way of seeing this is to look at the frequency responses:

$$H(\omega) = 1 + e^{-i5\omega} + e^{-i10\omega} \qquad F(\omega) = 1 + e^{-i\omega} + e^{-i2\omega}$$

so $\quad H(\omega) = F(5\omega)$

Alternatively, in the sample domain, we note that

$$h(n) = \begin{cases} f\left(\frac{n}{5}\right) & n \bmod 5 = 0 \\ 0 & \text{else} \end{cases}$$

which again leads to the relationship $H(\omega) = F(5\omega)$.

(f) (20 Points) Provide well-labeled plots of $|F(\omega)|$ and $\angle F(\omega)$, the magnitude and phase responses, respectively, of the subsystem F. Determine whether F is a low-pass filter, high-pass filter, band-pass filter, comb filter, notch filter, anti-notch filter, or some other type of filter.

Since RoC_F includes the unit circle

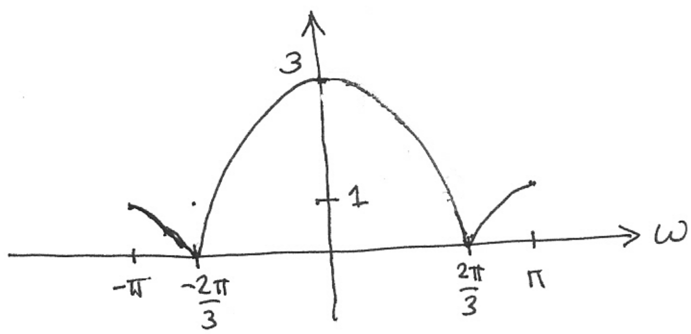
$$F(\omega) = \hat{F}(z)\Big|_{z=e^{i\omega}} = 1 + e^{-i\omega} + e^{-i2\omega}$$

$$= e^{-i\omega}\left(e^{i\omega} + 1 + e^{-i\omega}\right)$$

group these terms into $2\cos\omega$

$$F(\omega) = e^{-i\omega}\left(1 + 2\cos\omega\right)$$

So $|F(\omega)| = |1 + 2\cos\omega|$



$$\measuredangle F(\omega) = \underbrace{\measuredangle e^{-i\omega}}_{= -\omega} + \underbrace{\measuredangle(1 + 2\cos\omega)}_{= \begin{cases} 0 & \text{when } 1+2\cos\omega \geq 0 \\ \pm\pi & \text{when } 1-2\cos\omega < 0 \end{cases}}$$

$\measuredangle F(\omega)$