EECS 120 Signals & Systems — University of California, Berkeley: Fall 2007
Ramchandran — October 18, 2007

# Midterm Exam 1

| Last name | First name | SID |
|---|---|---|
| KEY | | |

*Rules.*

- You have two hours to complete this exam.
- There are 100 points for this exam.
- The exam is closed-book and closed-notes; calculators, computing and communication devices are *not* permitted.
- However, one handwritten and *not photocopied* double-sided sheet of notes is allowed.
- Moreover, you receive, together with the exam paper, copies of Tables 4.2 and 5.2 of the course textbook.
- No form of collaboration between the students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.

*Please read the following remarks carefully.*

- Show all work to get any partial credit.
- Take into account the points that may be earned for each problem when splitting your time between the problems.

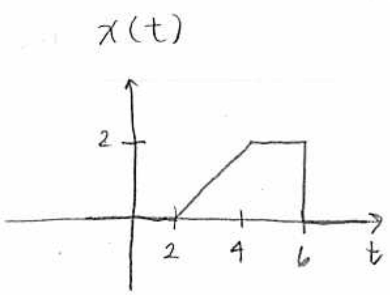
| Problem | Points earned | Points possible | Problem | Points earned | Points possible |
|---|---|---|---|---|---|
| Problem 1 | | 40 | Problem 2 | | 30 |
| Problem 3 | | 30 | | | |
| Total | | | | | 100 |

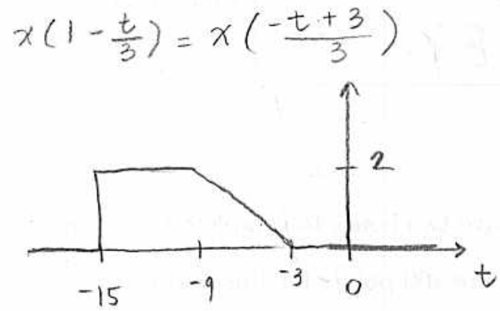
**Problem 1** *(Short questions.)*

*1. (a) 5 points*

$$\text{Given} \quad x(t) = \begin{cases} t-2, & 2 \le t < 4, \\ 2, & 4 \le t < 6, \\ 0, & \text{otherwise.} \end{cases}$$

Plot $x\left(1 - \frac{t}{3}\right)$. Label your axes clearly and carefully!




*1. (b) 5 points* For the following system, with input $x[n]$ and output $y[n]$, circle whether the statements are true or false.

$$y[n] = \sum_{k=-\infty}^{-2n} 3x[k]$$

T F the system is linear
T F the system is time-invariant
T F the system is memoryless
T F the system is stable
T F the system is causal

1. *(c) 7 points* An iron bar is heated to the temperature 300 degrees Celsius and placed in a room with ambient temperature $S$ degrees Celsius, where it is allowed to cool.

Every minute, the temperature of the bar decreases by an amount equal to 2% of the difference between the current temperature (at the start of that minute) and the ambient temperature. In the box below, write a difference equation for $T[n]$, the temperature of the bar after it has been in the room for $n$ minutes, and give any relevant initial conditions.

$$T[n] = T[n-1] - 0.02\,(T[n-1] - S)$$

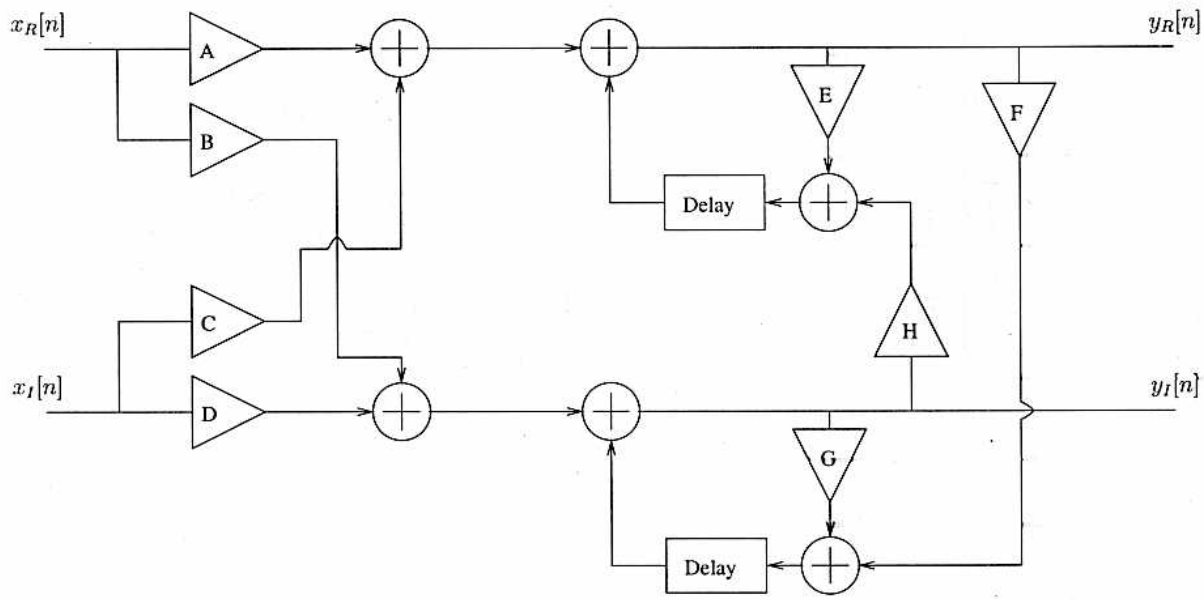
$$T[n] = 0.98\,T[n-1] + 0.02\,S$$

$$T[0] = 300$$

$$T[n] = 0.98\,T[n-1] + 0.02\,S$$

1. *(d) 4 points* Find the correct real gains in the block diagram below so that the input and output are related by the complex difference equation:

$$y[n] + (3-4j)\cdot y[n-1] = e^{-j\pi/2}x[n]$$




$$\begin{aligned}
y[n] &= -j\,x[n] - (3-4j)\,y[n-1] \\
&= -j\,(x_R[n] + j\,x_I[n]) - (3-4j)(y_R[n-1] + j\,y_I[n-1]) \\
&= -j\,x_R[n] + x_I[n] - (3y_R[n-1] + 3j\,y_I[n-1] - 4j\,y_R[n-1] + 4y_I[n-1]) \\
&= \underbrace{(x_I[n] - 3y_R[n-1] - 4y_I[n-1])}_{y_R[n]} + j\,\underbrace{(-x_R[n] - 3y_I[n-1] + 4y_R[n-1])}_{y_I[n]}
\end{aligned}$$

| | | |
|---|---|---|
| $A =$ 0 | $B =$ -1 | $C =$ 1 |
| $D =$ 0 | $E =$ -3 | $F =$ 4 |
| $G =$ -3 | $H =$ -4 | |

1. *(e) 6 points* A signal $x(t)$ is the input to an LTI system with impulse response $h(t) = \frac{\sin(500\pi t)}{\pi t}$. Which of the following signals could **not** be the output $y(t)$? (Circle your answer(s) and provide a brief explanation in the box below. No credit will be given for correct answers with incorrect reasoning.)

$y(t) = \cos(100\pi t)$

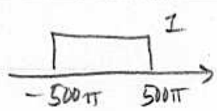
$y(t) = 12e^{j300\pi t}$

$y(t) = \sin(50\pi t) \cdot \cos(75\pi t)$

$y(t) = \sin(600\pi t)$

$y(t) = \sin(375\pi t)$

$H(jw) =$

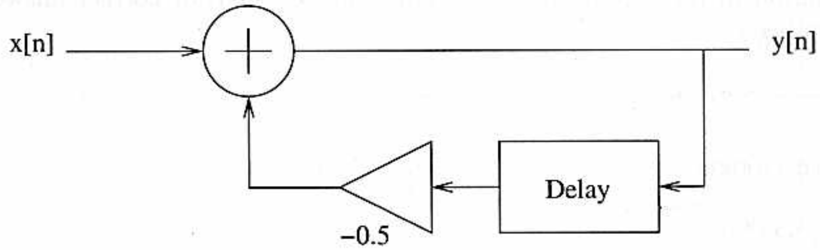



$(H(jw) = 0, |w| > 500\pi)$

If the system is LTI $Y(jw) = H(jw)X(jw)$, which means that $Y(jw) = 0$ $|w| > 500\pi$.

$y(t) = \sin(600\pi t)$ is the only signal that is not zero for $|w| > 500\pi$.

1. *(f) 6 points* Consider an LTI system with input $x[n]$ and output $y[n]$ that is implemented by the following block diagram




Find the frequency response $H(e^{j\Omega})$ of this system.

$$H(e^{j\Omega}) = \frac{1}{1+0.5e^{-j\Omega}}$$

$$y[n] = x[n] - 0.5\,y[n-1]$$

$$Y(e^{j\Omega}) = X(e^{j\Omega}) - 0.5e^{-j\Omega}Y(e^{j\omega})$$

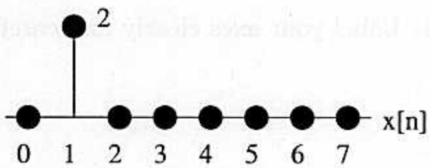
$$Y(e^{j\Omega})\left[1+0.5e^{-j\Omega}\right] = X(e^{j\Omega})$$

$$H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})} = \frac{1}{1+0.5e^{-j\Omega}}$$

1. *(g)* *7 points* A discrete-time LTI system, with input $x[n]$ and output $y[n]$, has frequency response

$$H(e^{j\Omega}) = \frac{1}{1 + 0.5e^{-j\Omega}}$$

The input signal $x[n]$ is periodic, with period $N = 8$. The following figure shows the value of $x[n]$ over the interval $0 \leq n \leq 7$.




Let $b_k$ denote the discrete-time Fourier series coefficients of $y[n]$. Compute the coefficient $b_4$.

$\boxed{b_4 = \quad -\frac{1}{2}}$

Since $x[n]$ is periodic using Fourier series we can rewrite $x[n]$ as a sum of exponentials.

$$x[n] \overset{FS}{\leftrightarrow} a_k \text{, where } a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{8}\sum_{n=0}^{7} x[n] e^{-jk\frac{2\pi}{8}n}$$

$$= \frac{1}{8}\, 2 e^{-jk\frac{2\pi}{8}(1)} = \frac{1}{4} e^{-jk\frac{\pi}{4}}$$

$$x[n] = \sum_{k=0}^{7} a_k e^{jk\frac{2\pi}{8}n}$$

Since, LTI we know $e^{j\Omega_0 n} \rightarrow \boxed{\substack{LTI \\ H(e^{j\Omega})}} \rightarrow H(e^{j\Omega_0}) e^{j\Omega_0 n}$

Thus,

$$y[n] = \sum_{k=0}^{7} \underbrace{a_k H(e^{jk\frac{2\pi}{8}})}_{b_K} e^{jk\frac{2\pi}{8}n} = \sum_{k=0}^{7} b_k e^{jk\frac{2\pi}{8}n}$$

$$b_4 = a_4 H(e^{j\frac{8\pi}{8}}) = a_4 H(e^{j\pi}) = \left[\frac{1}{4} e^{-j\frac{4\pi}{4}}\right]\left(\frac{1}{1+0.5e^{-j\pi}}\right)$$

$$= \left(-\frac{1}{4}\right)\left(\frac{1}{1-0.5}\right) = -\frac{1}{4} \cdot 2 = -\frac{1}{2}$$

7

**Problem 2** *(CTFT)*

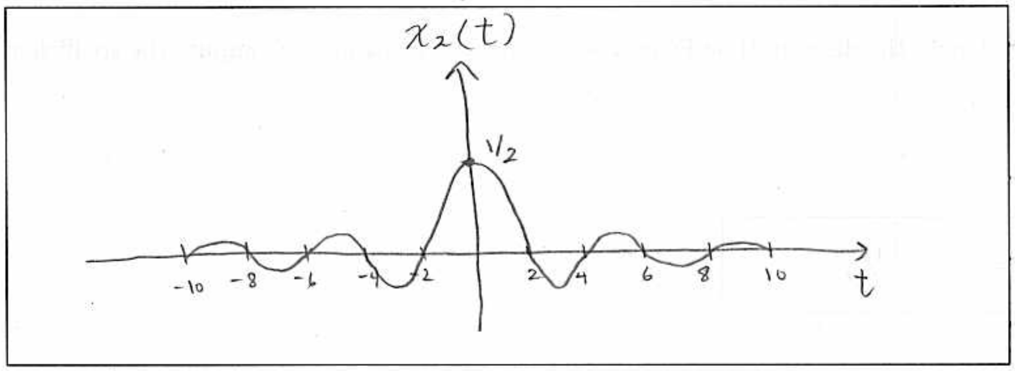
Consider the signal

$$x(t) = x_1(t) + x_2(t)$$

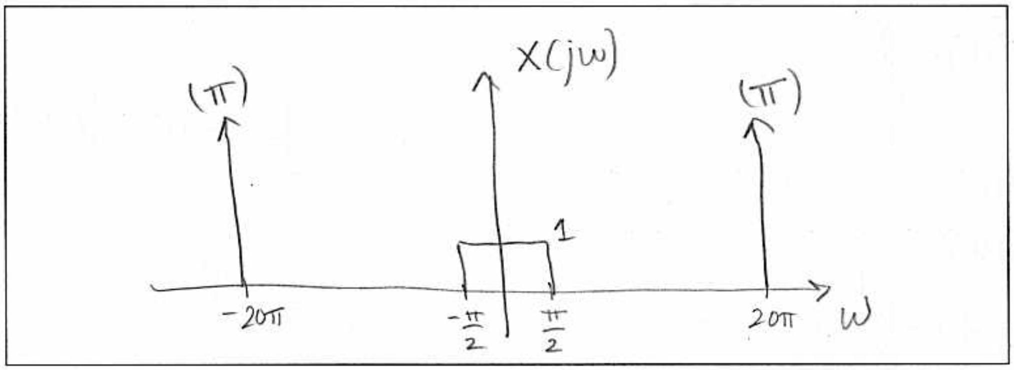
where

$$x_1(t) = \cos(20\pi t) \quad \text{and} \quad x_2(t) = \frac{\sin\left(\frac{\pi}{2}t\right)}{\pi t}$$

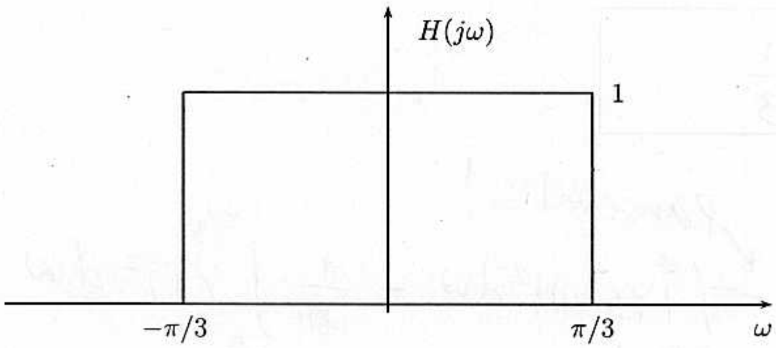
*2. (a) 6 points* Plot $x_2(t)$ from $-10 \le t \le 10$. Label your axes clearly and carefully!




*2. (b) 8 points* Plot the continuous-time Fourier transform of $x(t)$. Label your axes clearly and carefully!



2. *(c) 8 points* The signal $x(t)$ is now the input to an LTI system, whose frequency response $H(j\omega)$ is purely real and shown below.




Write an expression for the output of the LTI system, $y(t)$.

$Y(jw) = H(jw) \cdot X(jw) = H(jw)$

Thus, $y(t) = h(t)$

$$\boxed{y(t) = \frac{\sin(\frac{\pi}{3}t)}{\pi t}}$$

*2. (d) 8 points* Compute the energy $E_Y = \int_{-\infty}^{\infty} |y(t)|^2 dt$ of $y(t)$ from part (c).

$$\boxed{E_Y = \frac{1}{3}}$$

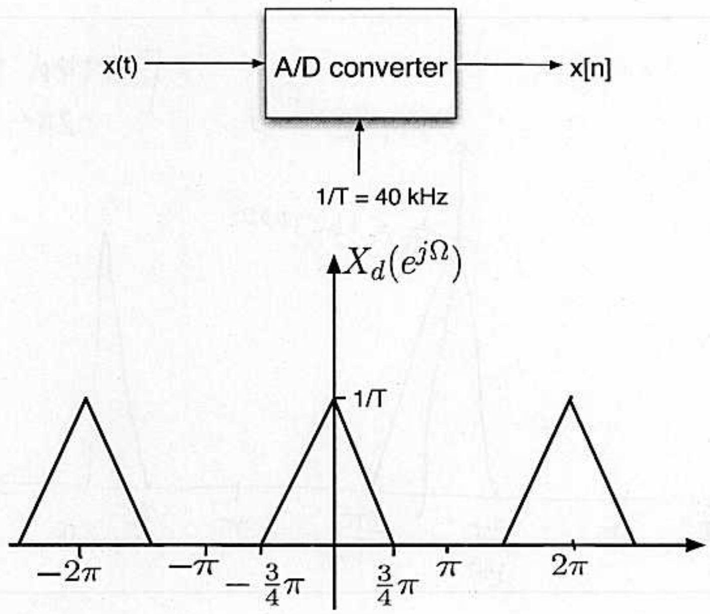
Parseval's !

$$\int_{-\infty}^{\infty} |y(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(jw)|^2 dw = \frac{1}{2\pi}\int_{-\frac{\pi}{3}}^{\pi/3} (1)^2 dw$$

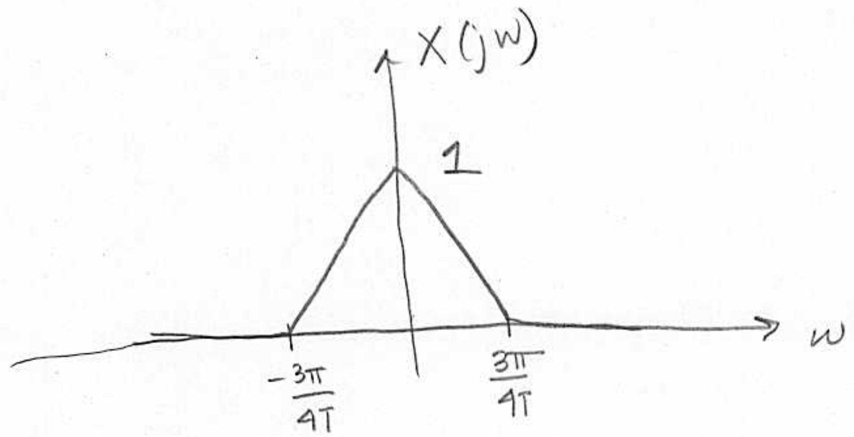
$$= \frac{1}{2\pi} \; w \Big|_{-\pi/3}^{\pi/3} = \frac{1}{2\pi}\left(\frac{\pi}{3} - \frac{-\pi}{3}\right)$$

$$= \frac{1}{2\pi}\left(\frac{2\pi}{3}\right) = \frac{1}{3}$$

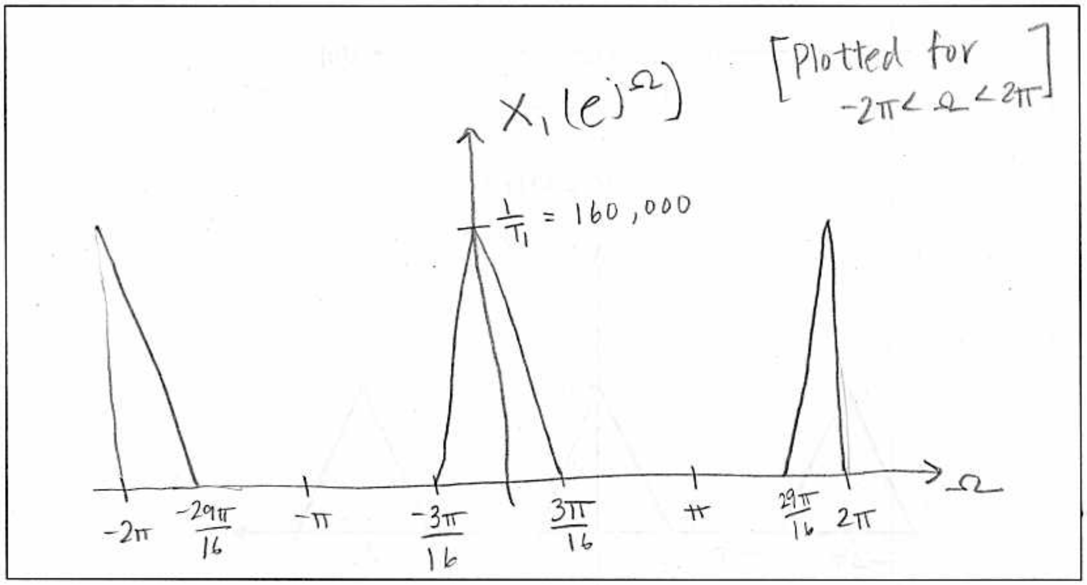
**Problem 3** *(Sampling)*



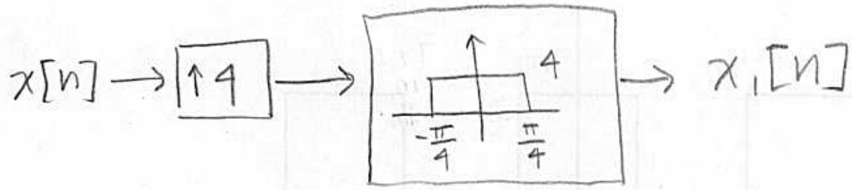

3. *(a) 5 points* x(t) is sampled above its Nyquist rate at $\frac{1}{T} = 40\ kHz$ to produce $x[n]$ whose spectrum, $X_d(e^{j\Omega})$, is shown in the figure above. Plot $X(j\omega)$, the spectrum of $x(t)$, clearly labeling your axes.

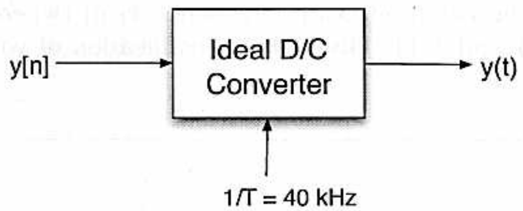


3. *(b) 5 points* For the same $x(t)$ as in part $(a)$, suppose the A/D converter is now operated at $\frac{1}{T_1} = 160\ kHz$ to produce $x_1[n]$. Plot the spectrum of $x_1[n]$.

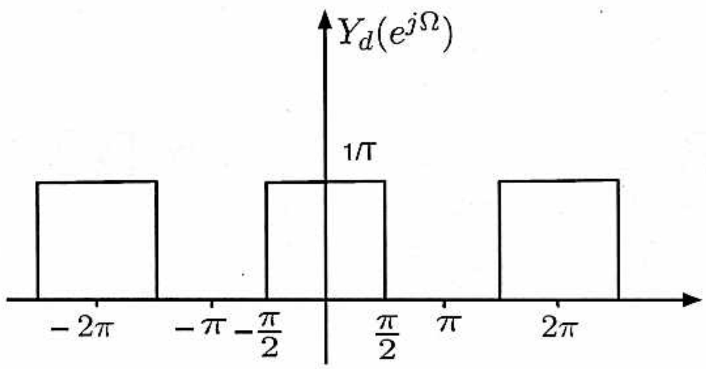


3. *(c) 6 points* Draw a discrete-time system with input $x[n]$ and output $x_1[n]$ (where $x[n]$ and $x_1[n]$ are the signals from parts $(a)$ and $(b)$). Give a brief justification of your answer to receive full credit.

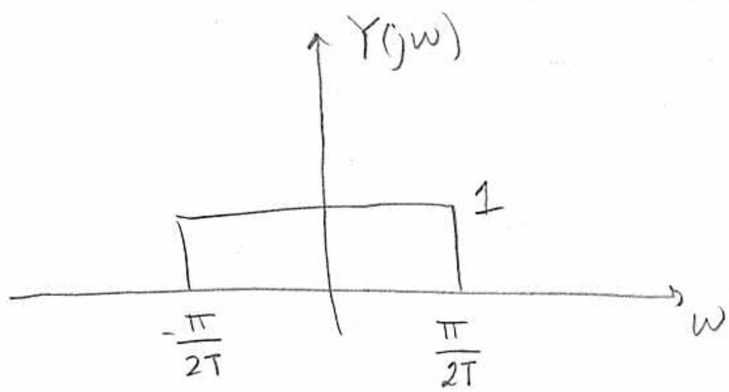


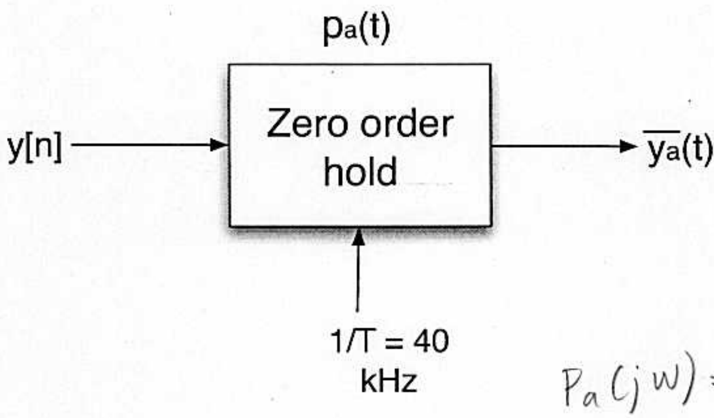



*3. (d) 6 points*

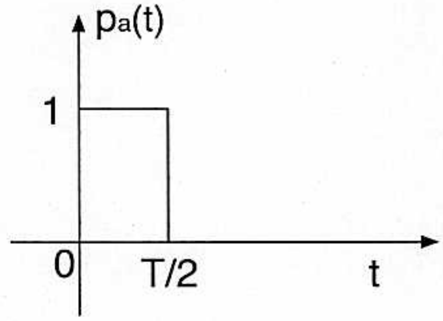



If $y[n]$ is the input to an ideal D/C running at $\frac{1}{T} = 40\ kHz$, and $y[n]$ has the spectrum shown in the figure above, plot $Y(j\omega)$, the spectrum of $y(t)$.






$$P_a(j\omega) = \frac{Te^{-j\omega T/4}}{2} \frac{\sin(\omega T/4)}{\omega T/4}$$




*3. (e) 8 points*

The signal $y[n]$ in part $(d)$ is the input to a Zero-Order Hold circuit characterized by $\bar{y}_a(t) = \sum_{n=-\infty}^{\infty} y[n]p_a(t-nT)$, where $p_a(t)$ is shown above. Note that this ZOH is holding for $\frac{T}{2}$ seconds, rather than the classical $T$ seconds. Plot the magnitude of the spectrum of $p(t)$ and the magnitude of the spectrum of $\bar{y}_a(t)$, both over the range $|\omega| < \frac{5\pi}{T}$.