CS 70 Spring 2008 — Discrete Mathematics for CS — David Wagner

Final Exam

PRINT your name: ______________________________, ______________________________
(last) (first)

SIGN your name: ____

PRINT your Unix account login: ____

Your section time (e.g., Tue 3pm): ____

Name of the person sitting to your left: ___________________________

Name of the person sitting to your right: ___________________________

You may consult any books, notes, or other paper-based inanimate objects available to you. Calculators and computers are not permitted. Please write your answers in the spaces provided in the test; in particular, we will not grade anything on the back of an exam page unless we are clearly told on the front of the page to look there.

You have 180 minutes. There are 8 questions, of varying credit (60 points total). The questions are of varying difficulty, so avoid spending too long on any one question.

Do not turn this page until your instructor tells you to do so.

Problem 1		Problem 6	
Problem 2		Problem 7	
Problem 3		Problem 8	
Problem 4			
Problem 5		Total	

Problem 1. [True or false] (8 points)

Circle TRUE or FALSE. Do not justify your answers on this problem.

(a) TRUE or FALSE: If $x, y \in \mathbb{N}$ and $2^{z-1} < y < x < 2^z$, then we can compute $x \times y$ using at most $O(z)$ bit operations.

(b) TRUE or FALSE: If $x, y \in \mathbb{N}$ and $2^{z-1} < y < x < 2^z$, then we can compute $\gcd(x, y)$ in at most $O(z^4)$ bit operations by using the Euclidean algorithm.

(c) TRUE or FALSE: 10 has a multiplicative inverse modulo 14.

(d) TRUE or FALSE: There are at most $O(n^2)$ different paths in any undirected graph with n vertices.

(e) TRUE or FALSE: If $X_1, \ldots, X_n$ are random variables and $X = X_1 + \cdots + X_n$, then $\mathbf{E}[X] = \mathbf{E}[X_1] + \cdots + \mathbf{E}[X_n]$ is guaranteed to hold, whether or not $X_1, \ldots, X_n$ are mutually independent.

(f) TRUE or FALSE: If $X_1, \ldots, X_n$ are random variables and $X = X_1 + \cdots + X_n$, then $\mathrm{Var}(X) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n)$ is guaranteed to hold, whether or not $X_1, \ldots, X_n$ are mutually independent.

Problem 2. [Grade this proof] (4 points)

Read this proof:

> **Theorem:** For all $n \geq 4$, we have $2^n < n!$.
>
> **Proof**: We will use proof by mathematical induction on n.
> *Base case:* If $n = 4$, then $2^4 = 16 < 24 = 4!$.
> *Inductive hypothesis:* Suppose we have $2^n < n!$ for some n with $n \geq 4$.
> *Inductive step:* We must show that $2^{n+1} < (n+1)!$, so we calculate:
>
> $$2^{n+1} = 2 \times 2^n < 2 \times n! < (n+1) \times n! = (n+1)!,$$
>
> where we used the inductive hypothesis as well as the fact that $2 < n+1$ if $n \geq 4$. Thus, we have shown that $2^n < n! \implies 2^{n+1} < (n+1)!$ for every $n \geq 4$, so by the principle of mathematical induction, we see that $2^n < n!$ holds for every $n \geq 4$. □

Now answer the question(s) below:

(a) TRUE or FALSE: This proof is valid.

(b) If you think this proof is invalid (i.e., if you answered False to part (a)), explain where the logical error in the proof lies.

Problem 3. [Grade this proof, too] (4 points)

Read this proof:

> **Theorem:** We have $2n \leq n^2 + 1$ for all $n \in \mathbb{N}$.
>
> **Proof**: We will prove this by simple induction on n. Let $P(n)$ denote the proposition that $2n \leq n^2 + 1$.
> *Base case:* If $n = 0$, then $2n = 0 \leq 0^2 + 1 = n^2 + 1$, so $P(0)$ is true.
> *Inductive hypothesis:* Assume $P(n)$ is true for some $n \in \mathbb{N}$. That is, we assume $2n \leq n^2 + 1$.
> *Inductive step:* We must show that $P(n+1)$ is true. Now
>
> $$2(n+1) = 2n + 2 \leq n^2 + 1 + 2 \leq (n+1)^2 + 1,$$
>
> where we have used the inductive hypothesis as well as the fact that $n^2 + 2 \leq (n+1)^2$. We see that $P(n) \implies P(n+1)$ holds for every $n \in \mathbb{N}$, so by the principle of mathematical induction, $P(n)$ is true for every $n \in \mathbb{N}$, and the theorem follows. $\square$

Now answer the question(s) below:

(a) TRUE or FALSE: This proof is valid.

(b) If you think this proof is invalid (i.e., if you answered False to part (a)), explain where the logical error in the proof lies.

Problem 4. [Short answer] (11 points)

Do not justify your answers to this problem. Do not show your work. Just show us your final answer.

(a) You flip a fair coin twice. Let A denote the event that at least one flip comes up heads and B the event that both flips come up heads. Calculate $\Pr[A]$.

(b) With the same notation as part (a), calculate $\Pr[A|B]$:

(c) With the same notation as part (a), calculate $\Pr[B|A]$:

(d) Flip a fair coin $2n$ times. Find a formula for the probability that we get exactly n heads and n tails.

(e) Define the sequence $a(0), a(1), a(2), \dots$ by $a(0) = 0$, $a(1) = 1$, and

$$a(n) = a(n-1) + a(n-2) + 1 \qquad \text{for } n \geq 2.$$

Remember that the Fibonacci numbers are defined by $F(0) = 0$, $F(1) = 1$, and

$$F(n) = F(n-1) + F(n-2) \qquad \text{for } n \geq 2.$$

Suppose we want to prove that $a(n) \leq F(n+2)$ for all $n \in \mathbb{N}$, using *simple* induction over n with some proposition $P(n)$. Show the definition of the proposition $P(n)$ that we should use. (Don't write down your proof here, just show us the proposition $P(n)$ you would use.)

(f) Call an infinite sequence $a_1, a_2, a_3, \dots$ of integers *good* if only finitely many of the a_is are nonzero. Multiple-choice: which of the following is true? Circle your choice.

(i) The set of all good sequences is finite.

(ii) The set of all good sequences is countably infinite.

(iii) The set of all good sequences is uncountably infinite.

Problem 5. [A Monty Hall Variant] (12 points)

Tired of hosting the same game year after year, Monty Hall decided to make some changes to his game. There are still three doors, but now one contains 1000 dollars, one contains 500 dollars, and one contains 0 dollars, with the order of the prizes randomly permuted. The contestant first selects a door. Then she has the choice of paying X dollars for Monty to open, among the two unchosen doors, the one that contains the smaller amount of money. If the contestant paid Monty, she then has the choice of switching to the other unopened door.

(a) Suppose the contestant refuses to pay Monty. In this case, what is the expected value of her prize?

(b) Suppose that the contestant decides to pay, and then Monty opens a door that contains \$500. Given this, what is the expected value of her prize if she switches?

Given this, what is the expected value of her prize if she sticks with her original door?

Multiple-choice: which of the following best describes her optimal strategy, in this situation, assuming she wants to maximize her profits? Circle your choice.

(i) She should stick with her initial door—that's strictly better than switching.

(ii) It doesn't matter whether she switches or sticks.

(iii) She should switch doors—that's strictly better than sticking.

(c) Now for a different scenario: Suppose that the contestant pays, and then Monty opens a door that contains \$0. Given this, what is the expected value of her prize if she switches?

Given this, what is the expected value of her prize if she sticks with her original door?

Multiple-choice: which of the following best describes her optimal strategy, in this situation, assuming she wants to maximize her profits? Circle your choice.

(i) She should stick with her initial door—that's strictly better than switching.

(ii) It doesn't matter whether she switches or sticks.

(iii) She should switch doors—that's strictly better than sticking.

(d) Now suppose a second contestant, Bob, decides in advance that he will always pay and always switch to the unopened door (no matter what he sees behind the door that Monty opens). What is the overall expected value of his prize, with this strategy?

(e) What is the most money Monty can charge for opening one of the two unchosen doors and still make it on average profitable for the contestant to pay Monty?

Problem 6. [Counting] (7 points)

For the purposes of this problem, a telephone number is an arbitrary sequence of 7 decimal digits. Do not justify your answer on this problem. You can leave your answer as an unevaluated expression.

(a) Call a telephone number *non-repetitious* if no pair of adjacent digits are the same. (For instance, 262-3762 is non-repetitious, but 523-3678 is repetitious.)

How many non-repetitious telephone numbers are there?

(b) Call a telephone *easy to dial* if it consists of only one or two different digits, in any sequence. (For instance, 661-6116 and 222-2222 and 888-8988 are easy to dial, but 121-1333 is not.)

How many easy-to-dial telephone numbers are there?

Problem 7. [Inversions] (6 points)

An *inversion* in a permutation $[a_1, a_2, \ldots, a_n]$ is a pair (a_i, a_j) such that $i < j$ but $a_i > a_j$. For example, in the list $[2,4,3,1]$ there are four inversions: $(2,1), (4,3), (4,1)$, and $(3,1)$.

Write down the numbers $1,2,\ldots,n$ in a random order, with all $n!$ orders equally likely. Let X denote the number of inversions in the resulting permutation.

(a) Compute $\mathbf{E}[X]$. Your answer should be a simple function of n. Show your work.

(b) Multiple-choice: With X defined as above, which of the following equations is true? Circle your choice.

(i) $\mathbf{E}[X^2] < \mathbf{E}[X]^2$.

(ii) $\mathbf{E}[X^2] = \mathbf{E}[X]^2$.

(iii) $\mathbf{E}[X^2] > \mathbf{E}[X]^2$.

Justify your answer *concisely*.

Problem 8. [A lame sorting routine] (8 points)

Consider the following bad algorithm for sorting an array A of n different integers:

1. Randomly shuffle A.
2. For $i = 0$ to $n - 2$, do the following:
3. If $A[i] > A[i+1]$, go back to step 1 and start over again.

(a) Let X be the total number of times the algorithm executes step 1. Compute $\mathbf{E}[X]$. Your answer should be a simple function of n. Show your work.

(b) Now suppose $n = 3$, and let Y denote the total number of times the algorithm executes the comparison in step 3. Calculate $\mathbf{E}[Y]$. Show your work.