CS174, Fall, 2004, Midterm 1, Professor Richard M. Karp

1. Consider two boxes, the first containing one black and one white marble, and the second containing two black and one white marble. A box is selected at random and a marble is drawn from it at random. Given that the marble is white what is the probability that the first box was selected?

2. In a permutation $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$, index $i$ is called a *cumulative maximum* if $\pi(i) = \max(\pi(1), \pi(2), \ldots, \pi(i))$. What is the expected number of cumulative maxima in a random permutation of $\{1, 2, \ldots, n\}$? Hint: What is the probability that $\pi(i)$ is cumulative maximum?

3. Let $E$ and $F$ be independent events in a finite sample space containing $n$ equally likely points. If $|E \cap F| = 8$, $|E \cap \overline{F}| = 6$, and $|\overline{E} \cap F| = 12$, what is $n$?

4. The nonnegative integer-valued random variable $X$ has expectation 500 and variance 100. What does Markov's Inequality imply about the probability that $X \geq 1000$? What does Chebyshev's inequality imply about the probability that $X$ lies in the interval [401,599]?

5. Random variable $X$ has the moment-generating function $e^{\lambda(e^t - 1)}$, where $\lambda$ is a constant. What are the expectation and variance of $X$?

6. Let $X$ and $Y$ be independent random variables.
Prove: $Var[X-Y]= Var[X] + Var[Y]$.

7. Let the random variable $X$ be the sum of $n$ indicator random variables $X_i$ (an indicator random variable, also known as a Bernoulli random variable, assumes only the values zero and one). The $X_i$ need not be independent.

(a) Using linearity of expectation, prove that $E[X^2] = \Sigma_{\text{i=1 to n}} E[X_i X]$.

(b) Using the conditional expectation identity, prove that

$E[X^2] = \Sigma_{\text{i=1 to n}} Pr(X_i = 1)E[X|X_i = 1]$.