CS 172: Computability and Complexity, Spring 2010 **S. A. Seshia & O. Etesami**

Final Exam

May 11, 2010

YOUR NAME:

Instructions:

This exam is *open-book, open-notes*. Please turn off electronic devices: cell phones, laptops, PDAs, etc.

You have a total of **180 minutes**. There are 6 questions worth a total of **150 points**. The questions vary in difficulty, so if you get stuck on any question, it might help to leave it for a while and try another one.

Answer each question in the space provided below the question. If you need more space, you can use the reverse side of that page. You may assume without proof any result that was proved in class or on a homework, but state your assumptions clearly. Descriptions of Turing machines can be in the form of Sipser's "high-level descriptions". Show your work in all proofs.

Do not turn this page until the instructor tells you to do so!

Problem 1		Problem 5	
Problem 2		Problem 6	
Problem 3			
Problem 4		Total	

Problem 1: [True or False, with justification] (40 points)

For each of the following 5 questions, state TRUE or FALSE. If TRUE, give a short proof. If FALSE, give a simple counterexample.

(a) The following language L is decidable:

$$L = \{\langle M\rangle \mid M \text{ is a TM and, for all inputs } w, M \text{ accepts } w \text{ within 10000 steps } \}$$

(b) If some regular language is NP-complete, then $P = NP$.

(c) If r and s are any two regular expressions, then $(r \cup s)^* = r^* \cup s^* \cup (rs)^*$.

Problem 1 (continued)

(d) If L is not a CFL, and $L' = L \setminus S$, for some finite set S, then L' is not a CFL.

(e) Recall the proof of the Cook-Levin theorem, where we used *legal* 2×3 windows. Suppose that the transition function δ of the TM N we are encoding is such that $\delta(q_1, a) = (q_2, b, R)$.

Then, the following 2×3 window is legal for machine N.

q_1	a	b
b	q_2	a

Problem 2: [Buggy Proofs] (15 points)

Each question below contains a false "proof" of some statement with *exactly one mistake*. For each part, identify the mistake in the reasoning and explain briefly why it is wrong. [NOTE: Answers that identify more than one mistake will be penalized! Note also that the statement itself may or may not be true.]

(a) Statement: Let L be regular. Then $L' = \{ww : w \in L\}$ is regular.
Proof: We recall first that the concatenation of two regular languages is regular. Since $L' = L \cdot L$, it follows that L' must also be regular.

(b) Statement: For every computable function $f : \mathbb{N} \to \mathbb{N}$, there exists a computable function $g : \mathbb{N} \to \mathbb{N}$ such that for every n we have $f(n) \leq g(n)$.
Proof: Assume, for the sake of arriving at a contradiction, that for some computable function f, for every computable function g, there exists an n such that $f(n) > g(n)$.

For each computable function g, let $n(g)$ be the smallest n such that $f(n) > g(n)$. Let n_0 be the maximum of $n(g)$ over all computable functions g, and assume this maximum is achieved for g_0, that is, $n(g_0) = n_0$.

Define a function g_1 that is equal to g_0 for all values except for n_0, and for n_0 we have $g_1(n_0) = f(n_0)$. This function g_1 is computable because 1) g_0 is computable and 2) g_1 differs from g_0 in only one point. By definition, $n(g_1) \geq n(g_0) + 1$, and this contradicts the fact that $n(g)$ was maximized at g_0.

Problem 3: (25 points)

(a) (10 points)
Describe how the following classes are related to each other using subsets and equality: (give the most precise relationship possible – i.e. give equality or strict containment where it holds; no justification needed)

NL, L, NPSPACE, PSPACE, EXPSPACE, P, NP, EXPTIME

(b) (15 points)
Prove or disprove that the following language is in PSPACE:
$N = \{\langle M \rangle | M$ is an NFA with $L(M) \neq \Sigma^*$ where Σ is the alphabet of $M\}$

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Problem 4: (25 points)

(a) (15 points)
Let A be a language that is neither empty nor Σ^*. Consider the following two sets of languages:
$L = \{B \mid B \leq_P A\}$ and $U = \{B \mid A \leq_P B\}$.

Prove that one of these sets is countable and the other one is uncountable.

(b) (10 points)

Let L be any language that is not Turing-recognizable. Define the language $L' = \{x|$ (x has odd length and $x \in L$) or (x has even length and $x \notin L)\}$.

Show that L' and its complement $\overline{L'}$ cannot both be Turing-recognizable.

Problem 5: (20 points)

Let L be the set of all strings with alphabet $\Sigma = \{a, b, c, d\}$ where the number of as is twice as much as the number of bs, and the number of cs is twice as much as the number of ds. Prove that L is not context-free.

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Problem 6: (25 points)

Consider the language ODD-CYCLE defined below:

$$\text{ODD-CYCLE} \quad = \quad \{\langle G\rangle \mid G \text{ is a directed graph with a simple directed cycle of odd length } \}$$

A cycle is *simple* if no vertex appears on the cycle twice. A cycle is *directed* if the cycle preserves the direction of the edges (i.e., all edges in the cycle are oriented in the same direction).

Show that ODD-CYCLE is NL-complete.

[Hint: Use a reduction from PATH. Don't forget to show that ODD-CYCLE is in NL.

Assume the following definition of PATH:

$$\text{PATH} \quad = \quad \{\langle G, s, t\rangle \mid G \text{ is a directed graph with a path from } s \text{ to } t \text{ } \}$$

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Thank you, and have a good summer!