Fall 2001
CS170: Efficient Algorithms and Intractable Problems
Professor Luca Trevisan
Midterm 2

Problem 1.

Provide the following information

Your name:

Your SID number:

Your section number (and/or your TA name):

Name of the person on your left (if any):

Name of the person on your right (if any):

Problem 2.

Consider a undirected graph G = (V, E) with nonnegative weights $w(i,j) \geq 0$ on its edges $(i, j) \in E$. Let s be a node in G. Assume you have computed the shortest paths from s, and minimum spanning tree of the graph. Suppose we change the weights on every edge by adding 1 to each of them. The new weights are $w'(i, j) = w(i, j) + 1$ for every $(i, j) \in E$.

(a) Would the minimum spanning tree change due to the change in weights? Give an example where it changes or prove that it cannot change.

(b) Would the shortest paths change due to the change in weights? Give and example where it changes or probe that it cannot change.

Problem 3.

There has been a lot of hype recently about Star Wars Episode II with the release of the newest theatrical trailer. For this problem, suppose you are managing the construction of billboards on Anakin Skywalker Memorial Highway, a heavily traveled stretch of road that runs west-east for M miles. The possible sites for billboards are given by numbers $x_1, x_2, ..., x_n$, each in the interval $[0, M]$ (specified by their position along the highway measured in miles from its western end). If you place a billboard at location x_i, you receive a revenue of $r_i > 0$.
You want to place billboards at a subset of the sites in $\{x_1, ...x_n\}$ so as to maximize your total revenue, subject to the following restrictions:

1. *Environmental Constraint.* You cannot build two billboards within less than 5 miles of one another on the highway.

2. *Boundary Constraint.* You cannot build a billboard within less than 5 miles of the western or eastern ends of the highway.

A subset of sites satisfying these two restrictions will be called *valid.*

Example: Suppose $M = 20$, $n = 4$

$\{x_1, x_2, x_3, x_4\} = \{6, 8,12, 14\}$

and

$\{r_1, r_2, r_3, r_4\} = \{5, 6, 5, 1\}$

Then the optimal solution would be to place billboards at x_1 and x_3 for a total revenue of 10.

Give an algorithm of running time polynomial in n, that takes an instances of this problem as input, and returns the maximum total revenue that can be obtained from any *valid* subset of sites.

Problem 4.

In a country with no antitrust laws, m software companies with values $W_1, ..., W_m$ are merged as follows. The two least valuable companies are merged , thus forming a new list of m - 1 companies. The value of the merged company is the sum of the values of the two companies that merged (we call this value the *volume of the merge*). This continues until only one company remains.

Let V be the total reported *volume of the merges.* For example if initially we had 4 companies of value (3,3,2,2), the merges yield

$$(3,3,2,2) \to (4,3,3) \to (6,4) \to (10)$$

and $V = 4 + 6 + 10 = 20$

Let us consider a situation with initial values $W_1, ..., W_m$, and let V be the total volume of the merges in the sequence of merges described above. Prove that merging the smallest pair at each step results in the minimum possible total volume after all companies are merged (i.e., $V \le V'$ where V is the result of our "algorithm" and V' is the result of any other algorithm for merging companies).

Problem 5.

Given a satisfiable system of linear inequalities

$$a_{11}x_1 + ... + a_{1n}x_n \le b_1$$
$$...$$
$$a_{m1}x_1 + ... + a_{mn}x_n \le b_m$$

we say that an inequality is *forced-equal* in the system if for every **x** that satisfies the system, the inequality is satisfied as an equality. (Equivalently, an inequality $\sum_i a_{ij}x_i \le b_j$ is *not* forced-equal in the system of there is an **x** that satisfies the whole system and such that in that inequality the left-hand side is strictly smaller than the right-hand side, that is $\sum a_{ij} x_i < b_j$.)

For example in

$$
\begin{aligned}
x_1 + x_2 &\le 2\\
-x_1 - x_2 &\le -2\\
x_1 &\le 1\\
-x_2 &\le 0
\end{aligned}
$$

the first two inequalities are forced-equal, while the third and fourth are not forced-equal. Observe that in any satisfiable system, there is always a solution where all inequalities that are not forced-equal have a left-hand side strictly smaller than the right-hand side. In the above example, we can set $x_1 = -1$ and $x_2 = 3$, so that we have $x_1 < 1$ and $-x_2 < 0$, as well as $x_1 + x_2 = 2$ and $-x_1 - x_2 = -2$.

Given a satisfiable system of linear inequalities, show how to use linear programming to determine which inequalities are forced-equal, and to find a solution where all inequalities that are not forced-equal have a left-hand side strictly smaller than the right-hand side.